# UNIT 18 EXERCISES 16-20

## COMPLEX

2009A

21. Let $p(x) = x^3 + ax^2 + bx + c$, where $a$, $b$, and $c$ are complex numbers. Suppose that

$$p(2009 + 9002\pi i) = p(2009) = p(9002) = 0.$$

What is the number of nonreal zeros of $x^{12} + ax^8 + bx^4 + c$?

**(A)** 4 **(B)** 6 **(C)** 8 **(D)** 10 **(E)** 12

2005B

22. A sequence of complex numbers $z_0, z_1, z_2, \ldots$ is defined by the rule

$$z_{n+1} = \frac{iz_n}{\overline{z_n}},$$

where $\overline{z_n}$ is the complex conjugate of $z_n$ and $i^2 = -1$. Suppose that $|z_0| = 1$ and $z_{2005} = 1$. How many possible values are there for $z_0$?

**(A)** 1 **(B)** 2 **(C)** 4 **(D)** 2005 **(E)** $2^{2005}$

2018A

22. The solutions to the equations $z^2 = 4 + 4\sqrt{15}i$ and $z^2 = 2 + 2\sqrt{3}i$, where $i = \sqrt{-1}$, form the vertices of a parallelogram in the complex plane. The area of this parallelogram can be written in the form $p\sqrt{q} - r\sqrt{s}$, where $p$, $q$, $r$, and $s$ are positive integers and neither $q$ nor $s$ is divisible by the square of any prime number. What is $p + q + r + s$?

**(A)** 20 **(B)** 21 **(C)** 22 **(D)** 23 **(E)** 24

2001

23. A polynomial of degree four with leading coefficient 1 and integer coefficients has two real zeros, both of which are integers. Which of the following can also be a zero of the polynomial?

**(A)** $\frac{1 + i\sqrt{11}}{2}$ **(B)** $\frac{1 + i}{2}$ **(C)** $\frac{1}{2} + i$ **(D)** $1 + \frac{i}{2}$ **(E)** $\frac{1 + i\sqrt{13}}{2}$

2008A

23. The solutions of the equation $z^4 + 4z^3 i - 6z^2 - 4zi - i = 0$ are the vertices of a convex polygon in the complex plane. What is the area of the polygon?

**(A)** $2^{\frac{5}{8}}$ **(B)** $2^{\frac{3}{4}}$ **(C)** $2$ **(D)** $2^{\frac{5}{4}}$ **(E)** $2^{\frac{3}{2}}$

2009B

23. A region $S$ in the complex plane is defined by

$$S = \{x + iy : -1 \le x \le 1, -1 \le y \le 1\}.$$

A complex number $z = x + iy$ is chosen uniformly at random from $S$. What is the probability that $(\frac{3}{4} + \frac{3}{4}i)z$ is also in $S$?

**(A)** $\dfrac{1}{2}$ **(B)** $\dfrac{2}{3}$ **(C)** $\dfrac{3}{4}$ **(D)** $\dfrac{7}{9}$ **(E)** $\dfrac{7}{8}$

2011A

23. Let $f(z) = \frac{z+a}{z+b}$ and $g(z) = f(f(z))$, where $a$ and $b$ are complex numbers. Suppose that $|a| = 1$ and $g(g(z)) = z$ for all $z$ for which $g(g(z))$ is defined. What is the difference between the largest and smallest possible values of $|b|$?

**(A)** $0$ **(B)** $\sqrt{2} - 1$ **(C)** $\sqrt{3} - 1$ **(D)** $1$ **(E)** $2$

2012B 23. Consider all polynomials of a complex variable, $P(z) = 4z^4 + az^3 + bz^2 + cz + d$, where $a, b, c$, and $d$ are integers, $0 \le d \le c \le b \le a \le 4$, and the polynomial has a zero $z_0$ with $|z_0| = 1$. What is the sum of all values $P(1)$ over all the polynomials with these properties?

**(A)** 84 **(B)** 92 **(C)** 100 **(D)** 108 **(E)** 120

2002A 24. Find the number of ordered pairs of real numbers $(a, b)$ such that $(a + bi)^{2002} = a - bi$.

**(A)** 1001 **(B)** 1002 **(C)** 2001 **(D)** 2002 **(E)** 2004

2011B 24. Let $P(z) = z^8 + (4\sqrt{3} + 6)z^4 - (4\sqrt{3} + 7)$. What is the minimum perimeter among all the 8-sided polygons in the complex plane whose vertices are precisely the zeros of $P(z)$?

**(A)** $4\sqrt{3} + 4$ **(B)** $8\sqrt{2}$ **(C)** $3\sqrt{2} + 3\sqrt{6}$ **(D)** $4\sqrt{2} + 4\sqrt{3}$ **(E)** $4\sqrt{3} + 6$

2013A 25. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = z^2 + iz + 1$. How many complex numbers $z$ are there such that $\text{Im}(z) > 0$ and both the real and the imaginary parts of $f(z)$ are integers with absolute value at most $10$?

**(A)** 399 **(B)** 401 **(C)** 413 **(D)** 431 **(E)** 441

2017A 25. The vertices $V$ of a centrally symmetric hexagon in the complex plane are given by

$$V = \left\{ \sqrt{2}i, -\sqrt{2}i, \frac{1}{\sqrt{8}}(1+i), \frac{1}{\sqrt{8}}(-1+i), \frac{1}{\sqrt{8}}(1-i), \frac{1}{\sqrt{8}}(-1-i) \right\}.$$

For each $j$, $1 \leq j \leq 12$, an element $z_j$ is chosen from $V$ at random, independently of the other choices. Let $P = \prod_{j=1}^{12} z_j$ be the product of the 12 numbers selected. What is the probability that $P = -1$?

**(A)** $\frac{5 \cdot 11}{3^{10}}$ **(B)** $\frac{5^2 \cdot 11}{2 \cdot 3^{10}}$ **(C)** $\frac{5 \cdot 11}{3^9}$ **(D)** $\frac{5 \cdot 7 \cdot 11}{2 \cdot 3^{10}}$

**(E)** $\frac{2^2 \cdot 5 \cdot 11}{3^{10}}$