UNIT 16 EXERCISES 16-20

FUNCTIONS

2011A

21. Let $f_1(x) = \sqrt{1-x}$, and for integers $n \geq 2$, let $f_n(x) = f_{n-1}(\sqrt{n^2 - x})$. If N is the largest value of n for which the domain of f_n is nonempty, the domain of f_N is $\{c\}$. What is $N + c$?

(A) -226 **(B)** -144 **(C)** -20 **(D)** 20 **(E)** 144

2007A

22. For each positive integer n, let $S(n)$ denote the sum of the digits of n. For how many values of n is $n + S(n) + S(S(n)) = 2007$?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

2009A

23. Functions f and g are quadratic, $g(x) = -f(100 - x)$, and the graph of g contains the vertex of the graph of f. The four x-intercepts on the two graphs have x-coordinates x_1, x_2, x_3, and x_4, in increasing order, and $x_3 - x_2 = 150$. The value of $x_4 - x_1$ is $m + n\sqrt{p}$, where m, n, and p are positive integers, and p is not divisible by the square of any prime. What is $m + n + p$?

(A) 602 **(B)** 652 **(C)** 702 **(D)** 752 **(E)** 802

2009A

24. The *tower function of twos* is defined recursively as follows: $T(1) = 2$ and $T(n+1) = 2^{T(n)}$ for $n \geq 1$. Let $A = (T(2009))^{T(2009)}$ and $B = (T(2009))^A$. What is the largest integer k such that

$$\underbrace{\log_2 \log_2 \log_2 \ldots \log_2 B}_{k \text{ times}}$$

is defined?

(A) 2009 **(B)** 2010 **(C)** 2011 **(D)** 2012 **(E)** 2013

2012B

24. Define the function f_1 on the positive integers by setting $f_1(1) = 1$ and if $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ is the prime factorization of $n > 1$, then

$$f_1(n) = (p_1 + 1)^{e_1 - 1}(p_2 + 1)^{e_2 - 1} \cdots (p_k + 1)^{e_k - 1}.$$

For every $m \geq 2$, let $f_m(n) = f_1(f_{m-1}(n))$. For how many N in the range $1 \leq N \leq 400$ is the sequence $(f_1(N), f_2(N), f_3(N), \ldots)$ unbounded?

Note: a sequence of positive numbers is unbounded if for every integer B, there is a member of the sequence greater than B.

(A) 15 **(B)** 16 **(C)** 17 **(D)** 18 **(E)** 19

2002B

25. Let $f(x) = x^2 + 6x + 1$, and let R denote the set of points (x, y) in the coordinate plane such that

$$f(x) + f(y) \leq 0 \quad \text{and} \quad f(x) - f(y) \leq 0.$$

The area of R is closest to

(A) 21 **(B)** 22 **(C)** 23 **(D)** 24 **(E)** 25

2003A

25. Let $f(x) = \sqrt{ax^2 + bx}$. For how many real values of a is there at least one positive value of b for which the domain of f and the range of f are the same set?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** infinitely many

2012A

25. Let $f(x) = |2\{x\} - 1|$ where $\{x\}$ denotes the fractional part of x. The number n is the smallest positive integer such that the equation

$$nf(xf(x)) = x$$

has at least 2012 real solutions x. What is n?

Note: the fractional part of x is a real number $y = \{x\}$, such that $0 \le y < 1$ and $x - y$ is an integer.

(A) 30 **(B)** 31 **(C)** 32 **(D)** 62 **(E)** 64