UNIT 14 EXERCISES 16-20

SEQUENCE

2002A

21. Consider the sequence of numbers: 4, 7, 1, 8, 9, 7, 6, ... For $n > 2$, the nth term of the sequence is the units digit of the sum of the two previous terms. Let S_n denote the sum of the first n terms of this sequence. The smallest value of n for which $S_n > 10,000$ is:

(A) 1992 **(B)** 1999 **(C)** 2001 **(D)** 2002 **(E)** 2004

2002B

21. For all positive integers n less than 2002, let

$$a_n = \begin{cases} 11, & \text{if } n \text{ is divisible by 13 and 14;} \\ 13, & \text{if } n \text{ is divisible by 14 and 11;} \\ 14, & \text{if } n \text{ is divisible by 11 and 13;} \\ 0, & \text{otherwise.} \end{cases}$$

Calculate $\sum_{n=1}^{2001} a_n$.

(A) 448 **(B)** 486 **(C)** 1560 **(D)** 2001 **(E)** 2002

2006A

23. Given a finite sequence $S = (a_1, a_2, \ldots, a_n)$ of n real numbers, let $A(S)$ be the sequence

$$\left(\frac{a_1 + a_2}{2}, \frac{a_2 + a_3}{2}, \ldots, \frac{a_{n-1} + a_n}{2}\right)$$

of $n-1$ real numbers. Define $A^1(S) = A(S)$ and, for each integer m, $2 \le m \le n-1$, define $A^m(S) = A(A^{m-1}(S))$. Suppose $x > 0$, and let $S = \left(1, x, x^2, \ldots, x^{100}\right)$. If $A^{100}(S) = \left(1/2^{50}\right)$, then what is x?

(A) $1 - \frac{\sqrt{2}}{2}$ **(B)** $\sqrt{2} - 1$ **(C)** $\frac{1}{2}$ **(D)** $2 - \sqrt{2}$ **(E)** $\frac{\sqrt{2}}{2}$

2001

25. Consider sequences of positive real numbers of the form x, 2000, y, ..., in which every term after the first is 1 less than the product of its two immediate neighbors. For how many different values of x does the term 2001 appear somewhere in the sequence?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** more than 4

2006B 25. A sequence a_1, a_2, $\ldots$ of non-negative integers is defined by the rule $a_{n+2} = |a_{n+1} - a_n|$ for $n \geq 1$. If $a_1 = 999$, $a_2 < 999$, and $a_{2006} = 1$, how many different values of a_2 are possible?

(A) 165 **(B)** 324 **(C)** 495 **(D)** 499 **(E)** 660

2008A 25. A sequence (a_1, b_1), (a_2, b_2), (a_3, b_3), $\ldots$ of points in the coordinate plane satisfies

$$(a_{n+1}, b_{n+1}) = (\sqrt{3}a_n - b_n, \sqrt{3}b_n + a_n) \quad \text{for} \quad n = 1, 2, 3, \ldots.$$

Suppose that $(a_{100}, b_{100}) = (2, 4)$. What is $a_1 + b_1$?

(A) $-\frac{1}{2^{97}}$ **(B)** $-\frac{1}{2^{99}}$ **(C)** 0 **(D)** $\frac{1}{2^{98}}$ **(E)** $\frac{1}{2^{96}}$

2009A 25. The first two terms of a sequence are $a_1 = 1$ and $a_2 = \frac{1}{\sqrt{3}}$. For $n \geq 1$,

$$a_{n+2} = \frac{a_n + a_{n+1}}{1 - a_n a_{n+1}}.$$

What is $|a_{2009}|$?

(A) 0 **(B)** $2 - \sqrt{3}$ **(C)** $\frac{1}{\sqrt{3}}$ **(D)** 1 **(E)** $2 + \sqrt{3}$

2016B 25. The sequence (a_n) is defined recursively by $a_0 = 1$, $a_1 = \sqrt[19]{2}$, and $a_n = a_{n-1}a_{n-2}^2$ for $n \geq 2$. What is the smallest positive integer k such that the product $a_1 a_2 \cdots a_k$ is an integer?

(A) 17 **(B)** 18 **(C)** 19 **(D)** 20 **(E)** 21

1999 28. Let $x_1, x_2, \ldots, x_n$ be a sequence of integers such that
(i) $-1 \leq x_i \leq 2$, for $i = 1, 2, 3, \ldots, n$;
(ii) $x_1 + x_2 + \cdots + x_n = 19$; and
(iii) $x_1^2 + x_2^2 + \cdots + x_n^2 = 99$.
Let m and M be the minimal and maximal possible values of $x_1^3 + x_2^3 + \cdots + x_n^3$, respectively. Then $M/m =$

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7