UNIT 13 EXERCISES 16-20

NUMBER THEO/ ALGEBRA WORD

2005A

2006A

2007B

2017A

21. A set S is constructed as follows. To begin, $S = \{0, 10\}$. Repeatedly, as long as possible, if x is an integer root of some polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ for some $n \geq 1$, all of whose coefficients a_i are elements of S, then x is put into S. When no more elements can be added to S, how many elements does S have?

(A) 4 **(B)** 5 **(C)** 7 **(D)** 9 **(E)** 11

2015A

2016B 22. For a certain positive integer n less than 1000, the decimal equivalent of $\frac{1}{n}$ is $0.\overline{abcdef}$, a repeating decimal of period 6, and the decimal equivalent of $\frac{1}{n+6}$ is $0.\overline{wxyz}$, a repeating decimal of period 4. In which interval does n lie?

(A) $[1, 200]$ **(B)** $[201, 400]$ **(C)** $[401, 600]$ **(D)** $[601, 800]$ **(E)** $[801, 999]$

2014A 23. The fraction

$$\frac{1}{99^2} = 0.\overline{b_{n-1}b_{n-2}\ldots b_2b_1b_0},$$

where n is the length of the period of the repeating decimal expansion. What is the sum $b_0 + b_1 + \cdots + b_{n-1}$?

(A) 874 **(B)** 883 **(C)** 887 **(D)** 891 **(E)** 892

2012A

24. Let $\{a_k\}_{k=1}^{2011}$ be the sequence of real numbers defined by

$$a_1 = 0.201, \quad a_2 = (0.2011)^{a_1}, \quad a_3 = (0.20101)^{a_2}, \quad a_4 = (0.201011)^{a_3},$$

and more generally

$$a_k = \begin{cases} (0.\underbrace{20101\ldots0101}_{k+2 \text{ digits}})^{a_{k-1}}, & \text{if } k \text{ is odd,} \\ (0.\underbrace{20101\ldots01011}_{k+2 \text{ digits}})^{a_{k-1}}, & \text{if } k \text{ is even.} \end{cases}$$

Rearranging the numbers in the sequence $\{a_k\}_{k=1}^{2011}$ in decreasing order produces a new sequence $\{b_k\}_{k=1}^{2011}$. What is the sum of all the integers k, $1 \le k \le 2011$, such that $a_k = b_k$?

(A) 671 **(B)** 1006 **(C)** 1341 **(D)** 2011 **(E)** 2012

1999

25. There are unique integers $a_2, a_3, a_4, a_5, a_6, a_7$ such that

$$\frac{5}{7} = \frac{a_2}{2!} + \frac{a_3}{3!} + \frac{a_4}{4!} + \frac{a_5}{5!} + \frac{a_6}{6!} + \frac{a_7}{7!},$$

where $0 \le a_i < i$ for $i = 2, 3, \ldots, 7$. Find $a_2 + a_3 + a_4 + a_5 + a_6 + a_7$.

(A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

2004B

25. Given that 2^{2004} is a 604-digit number whose first digit is 1, how many elements of the set $S = \{2^0, 2^1, 2^2, \ldots, 2^{2003}\}$ have a first digit of 4?

(A) 194 **(B)** 195 **(C)** 196 **(D)** 197 **(E)** 198

2006A

25. How many non-empty subsets S of $\{1, 2, 3, \ldots, 15\}$ have the following two properties?

(1) No two consecutive integers belong to S.

(2) If S contains k elements, then S contains no number less than k.

(A) 277 **(B)** 311 **(C)** 376 **(D)** 377 **(E)** 405

2016A

25. Let k be a positive integer. Bernardo and Silvia take turns writing and erasing numbers on a blackboard as follows: Bernardo starts by writing the smallest perfect square with $k+1$ digits. Every time Bernardo writes a number, Silvia erases the last k digits of it. Bernardo then writes the next perfect square, Silvia erases the last k digits of it, and this process continues until the last two numbers that remain on the board differ by at least 2. Let $f(k)$ be the smallest positive integer not written on the board. For example, if $k = 1$, then the numbers that Bernardo writes are $16, 25, 36, 49$, and 64, and the numbers showing on the board after Silvia erases are $1, 2, 3, 4$, and 6, and thus $f(1) = 5$. What is the sum of the digits of $f(2) + f(4) + f(6) + \cdots + f(2016)$?

(A) 7986 **(B)** 8002 **(C)** 8030 **(D)** 8048 **(E)** 8064

2018A 25. For a positive integer n and nonzero digits a, b, and c, let A_n be the n-digit integer each of whose digits is equal to a; let B_n be the n-digit integer each of whose digits is equal to b; and let C_n be the $2n$-digit (not n-digit) integer each of whose digits is equal to c. What is the greatest possible value of $a + b + c$ for which there are at least two values of n such that $C_n - B_n = A_n^2$?

(A) 12 **(B)** 14 **(C)** 16 **(D)** 18 **(E)** 20

1999 30. The number of ordered pairs of integers (m, n) for which $mn \geq 0$ and

$$m^3 + n^3 + 99mn = 33^3$$

is equal to

(A) 2 **(B)** 3 **(C)** 33 **(D)** 35 **(E)** 99