UNIT 10 EXERCISES 16-20

ALGEBRA

2002B

22. For all integers n greater than 1, define $a_n = \dfrac{1}{\log_n 2002}$. Let $b = a_2+a_3+a_4+a_5$ and $c = a_{10} + a_{11} + a_{12} + a_{13} + a_{14}$. Then $b - c$ equals

(A) -2 **(B)** -1 **(C)** $\dfrac{1}{2002}$ **(D)** $\dfrac{1}{1001}$ **(E)** $\dfrac{1}{2}$

2006B 22. Suppose a, b, and c are positive integers with $a+b+c = 2006$, and $a!b!c! = m \cdot 10^n$, where m and n are integers and m is not divisible by 10. What is the smallest possible value of n?

(A) 489 **(B)** 492 **(C)** 495 **(D)** 498 **(E)** 501

2010B 22. Let $ABCD$ be a cyclic quadrilateral. The side lengths of $ABCD$ are distinct integers less than 15 such that $BC \cdot CD = AB \cdot DA$. What is the largest possible value of BD?

(A) $\sqrt{\frac{325}{2}}$ **(B)** $\sqrt{185}$ **(C)** $\sqrt{\frac{389}{2}}$ **(D)** $\sqrt{\frac{425}{2}}$ **(E)** $\sqrt{\frac{533}{2}}$

2012A 22. Distinct planes $p_1, p_2, \ldots, p_k$ intersect the interior of a cube Q. Let S be the union of the faces of Q and let $P = \bigcup_{j=1}^{k} p_j$. The intersection of P and S consists of the union of all segments joining the midpoints of every pair of edges belonging to the same face of Q. What is the difference between the maximum and the minimum possible values of k?

(A) 8 **(B)** 12 **(C)** 20 **(D)** 23 **(E)** 24

2014A 22. The number 5^{867} is between 2^{2013} and 2^{2014}. How many pairs of integers (m, n) are there such that $1 \leq m \leq 2012$ and

$$5^n < 2^m < 2^{m+2} < 5^{n+1} ?$$

(A) 278 **(B)** 279 **(C)** 280 **(D)** 281 **(E)** 282

2003A 23. How many perfect squares are divisors of the product $1! \cdot 2! \cdot 3! \cdots 9!$?

(A) 504 **(B)** 672 **(C)** 864 **(D)** 936 **(E)** 1008

2004A 23. A polynomial

$$P(x) = c_{2004}x^{2004} + c_{2003}x^{2003} + \cdots + c_1x + c_0$$

has real coefficients with $c_{2004} \neq 0$ and 2004 distinct complex zeros $z_k = a_k + b_k i$, $1 \leq k \leq 2004$ with a_k and b_k real, $a_1 = b_1 = 0$, and

$$\sum_{k=1}^{2004} a_k = \sum_{k=1}^{2004} b_k.$$

Which of the following quantities can be a nonzero number?

(A) c_0 **(B)** c_{2003} **(C)** $b_2b_3 \dots b_{2004}$ **(D)** $\sum_{k=1}^{2004} a_k$ **(E)** $\sum_{k=1}^{2004} c_k$

2008B 23. The sum of the base-10 logarithms of the divisors of 10^n is 792. What is n?

(A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15

2010A 23. The number obtained from the last two nonzero digits of 90! is equal to n. What is n?

(A) 12 **(B)** 32 **(C)** 48 **(D)** 52 **(E)** 68

2003A 24. If $a \geq b > 1$, what is the largest possible value of $\log_a(a/b) + \log_b(b/a)$?

(A) -2 **(B)** 0 **(C)** 2 **(D)** 3 **(E)** 4

2006A 24. The expression

$$(x+y+z)^{2006}+(x-y-z)^{2006}$$

is simplified by expanding it and combining like terms. How many terms are in the simplified expression?

(A) 6018 **(B)** 671,676 **(C)** 1,007,514 **(D)** 1,008,016 **(E)** 2,015,028

2007B 24. How many pairs of positive integers (a, b) are there such that $\gcd(a, b) = 1$ and

$$\frac{a}{b}+\frac{14b}{9a}$$

is an integer?

(A) 4 **(B)** 6 **(C)** 9 **(D)** 12 **(E)** infinitely many

2010B 24. The set of real numbers x for which

$$\frac{1}{x-2009}+\frac{1}{x-2010}+\frac{1}{x-2011}\geq 1$$

is the union of intervals of the form $a < x \leq b$. What is the sum of the lengths of these intervals?

(A) $\frac{1003}{335}$ **(B)** $\frac{1004}{335}$ **(C)** 3 **(D)** $\frac{403}{134}$ **(E)** $\frac{202}{67}$

2018B

24. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. How many real numbers x satisfy the equation $x^2 + 10{,}000\lfloor x \rfloor = 10{,}000x$?

(A) 197 **(B)** 198 **(C)** 199 **(D)** 200 **(E)** 201

2004A

25. For each integer $n \geq 4$, let a_n denote the base-n number $0.\overline{133}_n$. The product $a_4 a_5 \dots a_{99}$ can be expressed as $\frac{m}{n!}$, where m and n are positive integers and n is as small as possible. What is the value of m?

(A) 98 **(B)** 101 **(C)** 132 **(D)** 798 **(E)** 962

2010B

25. For every integer $n \geq 2$, let $\text{pow}(n)$ be the largest power of the largest prime that divides n. For example $\text{pow}(144) = \text{pow}(2^4 \cdot 3^2) = 3^2$. What is the largest integer m such that 2010^m divides

$$\prod_{n=2}^{5300} \text{pow}(n)?$$

(A) 74 **(B)** 75 **(C)** 76 **(D)** 77 **(E)** 78