UNIT 20 QUESTIONS 16-20

FUNCTIONS

2004A 17. Let f be a function with the following properties:

(i) $f(1) = 1$, and

(ii) $f(2n) = n \cdot f(n)$ for any positive integer n.

What is the value of $f(2^{100})$?

(A) 1 **(B)** 2^{99} **(C)** 2^{100} **(D)** 2^{4950} **(E)** 2^{9999}

2011B 17. Let $f(x) = 10^{10x}$, $g(x) = \log_{10}\left(\frac{x}{10}\right)$, $h_1(x) = g(f(x))$, and $h_n(x) = h_1(h_{n-1}(x))$ for integers $n \geq 2$. What is the sum of the digits of $h_{2011}(1)$?

(A) 16,081 **(B)** 16,089 **(C)** 18,089 **(D)** 18,098 **(E)** 18,099

2006A 18. The function f has the property that for each real number x in its domain, $1/x$ is also in its domain and

$$f(x) + f\left(\frac{1}{x}\right) = x.$$

What is the largest set of real numbers that can be in the domain of f?

(A) $\{x \mid x \neq 0\}$ **(B)** $\{x \mid x < 0\}$ **(C)** $\{x \mid x > 0\}$

(D) $\{x \mid x \neq -1 \text{ and } x \neq 0 \text{ and } x \neq 1\}$ **(E)** $\{-1, 1\}$

2015B 18. For every composite positive integer n, define $r(n)$ to be the sum of the factors in the prime factorization of n. For example, $r(50) = 12$ because the prime factorization of 50 is $2 \cdot 5^2$, and $2 + 5 + 5 = 12$. What is the range of the function r, $\{r(n) : n \text{ is a composite positive integer}\}$?

(A) the set of positive integers

(B) the set of composite positive integers

(C) the set of even positive integers

(D) the set of integers greater than 3

(E) the set of integers greater than 4

2018B 18. A function f is defined recursively by $f(1) = f(2) = 1$ and

$$f(n) = f(n-1) - f(n-2) + n$$

for all integers $n \geq 3$. What is $f(2018)$?

(A) 2016 **(B)** 2017 **(C)** 2018 **(D)** 2019 **(E)** 2020

2005A 20. For each x in $[0, 1]$, define

$$f(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2 - 2x, & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Let $f^{[2]}(x) = f(f(x))$, and $f^{[n+1]}(x) = f^{[n]}(f(x))$ for each integer $n \geq 2$. For how many values of x in $[0, 1]$ is $f^{[2005]}(x) = 1/2$?

(A) 0 **(B)** 2005 **(C)** 4010 **(D)** 2005^2 **(E)** 2^{2005}

2015B 20. For every positive integer n, let $\text{mod}_5(n)$ be the remainder obtained when n is divided by 5. Define a function $f : \{0,1,2,3,\ldots\} \times \{0,1,2,3,4\} \to \{0,1,2,3,4\}$ recursively as follows:

$$f(i,j) = \begin{cases} \text{mod}_5(j+1) & \text{if } i = 0 \text{ and } 0 \le j \le 4, \\ f(i-1,1) & \text{if } i \ge 1 \text{ and } j = 0, \text{ and} \\ f(i-1, f(i,j-1)) & \text{if } i \ge 1 \text{ and } 1 \le j \le 4. \end{cases}$$

What is $f(2015,2)$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

2016A 20. A binary operation $\diamondsuit$ has the properties that $a \diamondsuit (b \diamondsuit c) = (a \diamondsuit b) \cdot c$ and that $a \diamondsuit a = 1$ for all nonzero real numbers a, b, and c. (Here the dot $\cdot$ represents the usual multiplication operation.) The solution to the equation $2016 \diamondsuit (6 \diamondsuit x) = 100$ can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p + q$?

(A) 109 **(B)** 201 **(C)** 301 **(D)** 3049 **(E)** 33,601