UNIT 21 EXERCISES 6-10

SEQUENCE

2003B

6. (B) Let the sequence be denoted $a, ar, ar^2, ar^3, \ldots$, with $ar = 2$ and $ar^3 = 6$. Then $r^2 = 3$ and $r = \sqrt{3}$ or $r = -\sqrt{3}$. Therefore $a = \frac{2\sqrt{3}}{3}$ or $a = -\frac{2\sqrt{3}}{3}$.

2016A 6. **Answer (D):** There are

$$1 + 2 + \cdots + N = \frac{N(N+1)}{2}$$

coins in the array. Therefore $N(N+1) = 2 \cdot 2016 = 4032$. Because $N(N+1) \approx N^2$, it follows that $N \approx \sqrt{4032} \approx \sqrt{2^{12}} = 2^6 = 64$. Indeed, $63 \cdot 64 = 4032$, so $N = 63$ and the sum of the digits of N is 9.

2007A 7. **Answer (C):** Let D be the difference between consecutive terms of the sequence. Then $a = c - 2D$, $b = c - D$, $d = c + D$, and $e = c + 2D$, so

$$a + b + c + d + e = (c - 2D) + (c - D) + c + (c + D) + (c + 2D) = 5c.$$

Thus $5c = 30$, so $c = 6$.

To see that the values of the other terms cannot be found, note that the sequences $4, 5, 6, 7, 8$ and $10, 8, 6, 4, 2$ both satisfy the given conditions.

2009A 7. **Answer (B):** Because the difference between consecutive terms is constant,

$$(5x - 11) - (2x - 3) = (3x + 1) - (5x - 11).$$

Therefore $x = 4$, and the first three terms are 5, 9, and 13. Thus the difference between consecutive terms is 4. The nth term is $2009 = 5 + (n - 1) \cdot 4$, and it follows that $n = 502$.

2012A

7. **Answer (C):** Let a be the initial term and d the common difference for the arithmetic sequence. Then the sum of the degree measures of the central angles is

$$a + (a + d) + \cdots + (a + 11d) = 12a + 66d = 360,$$

so $2a + 11d = 60$. Letting $d = 4$ yields the smallest possible positive integer value for a, namely $a = 8$.

2013A

7. **Answer (C):**

Note that $110 = S_9 = S_7 + S_8 = 42 + S_8$, so $S_8 = 110 - 42 = 68$. Thus $68 = S_8 = S_6 + S_7 = S_6 + 42$, so $S_6 = 68 - 42 = 26$. Similarly, $S_5 = 42 - 26 = 16$, and $S_4 = 26 - 16 = 10$.

2014A

7. **Answer (A):** Each term in a geometric progression is r times the preceding term. The ratio is

$$r = \frac{3^{\frac{1}{3}}}{3^{\frac{1}{2}}} = 3^{\frac{1}{3} - \frac{1}{2}} = 3^{-\frac{1}{6}}.$$

Thus the third term is correctly given as $r \cdot 3^{\frac{1}{3}} = 3^{-\frac{1}{6}} \cdot 3^{\frac{1}{3}} = 3^{\frac{1}{6}}$, and the fourth term is $r \cdot 3^{\frac{1}{6}} = 3^{-\frac{1}{6}} \cdot 3^{\frac{1}{6}} = 3^0 = 1$.

1999

8. **(D)** Let w and $2w$ denote the ages of Walter and his grandmother, respectively, at the end of 1994. Then their respective years of birth are $1994 - w$ and $1994 - 2w$. Hence $(1994 - w) + (1994 - 2w) = 3838$, and it follows that $w = 50$ and Walter's age at the end of 1999 will be 55.

2004B

8. **(D)** If there are n rows in the display, the bottom row contains $2n - 1$ cans. The total number of cans is therefore the sum of the arithmetic series

$$1 + 3 + 5 + \cdots + (2n - 1),$$

which is

$$\frac{n}{2}[(2n - 1) + 1] = n^2.$$

Thus $n^2 = 100$, so $n = 10$.

2011A

8. **Answer (A):** The only parts of the track that are longer walking on the outside edge rather than the inside edge are the two semicircular portions. If the radius of the inner semicircle is r, then the difference in the lengths of the two paths is $2\pi(r + 6) - 2\pi r = 12\pi$. Let x be her speed in meters per second. Then $36x = 12\pi$, and $x = \frac{\pi}{3}$.

2002B 9. **(C)** We have $b = a + r$, $c = a + 2r$, and $d = a + 3r$, where r is a positive real number. Also, $b^2 = ad$ yields $(a + r)^2 = a(a + 3r)$, or $r^2 = ar$. It follows that $r = a$ and $d = a + 3a = 4a$. Hence $\dfrac{a}{d} = \dfrac{1}{4}$.

2018B

9. **Answer (E):** Note that the sum of the first 100 positive integers is $\frac{1}{2} \cdot 100 \cdot 101 = 5050$. Then

$$\begin{aligned}\sum_{i=1}^{100}\sum_{j=1}^{100}(i+j) &= \sum_{i=1}^{100}\sum_{j=1}^{100} i + \sum_{i=1}^{100}\sum_{j=1}^{100} j \\ &= \sum_{j=1}^{100}\sum_{i=1}^{100} i + \sum_{i=1}^{100}\sum_{j=1}^{100} j \\ &= 100\sum_{i=1}^{100} i + 100\sum_{j=1}^{100} j \\ &= 100(5050+5050) \\ &= 1{,}010{,}000.\end{aligned}$$

OR

Note that the sum of the first 100 positive integers is $\frac{1}{2} \cdot 100 \cdot 101 = 5050$. Then

$$\begin{aligned}\sum_{i=1}^{100}\sum_{j=1}^{100}(i+j) &= \sum_{i=1}^{100}\big((i+1)+(i+2)+\cdots+(i+100)\big) \\ &= \sum_{i=1}^{100}(100i+5050) \\ &= 100 \cdot 5050 + 100 \cdot 5050 \\ &= 1{,}010{,}000.\end{aligned}$$

OR

The sum contains 10,000 terms, and the average value of both i and j is $\frac{101}{2}$, so the sum is equal to

$$10{,}000\left(\frac{101}{2}+\frac{101}{2}\right) = 1{,}010{,}000.$$

2005B 10. **(E)** The sequence begins 2005, 133, 55, 250, 133, ... Thus after the initial term 2005, the sequence repeats the cycle 133, 55, 250. Because $2005 = 1 + 3 \cdot 668$, the 2005$^{\text{th}}$ term is the same as the last term of the repeating cycle, 250.

2010A 10. **Answer (A):** Consecutive terms in an arithmetic sequence have a common difference d. Thus $(3p + q) - (3p - q) = 2q = d$. Further, the second term is equal to $p + d$, so $p + d = 9$, and the third term is equal to $p + 2d$, so $p + 2d = 3p - q$. These three equations form a system that can be solved to yield $p = 5$, $q = 2$, and $d = 4$. Therefore the 2010$^{\text{th}}$ term of the sequence is $p + 2009d = 5 + 2009 \cdot 4 = 8041$.