UNIT 18 EXERCISES 6-10

WORD ALGEBRA

1999 6. **(D)** Note that

$$2^{1999} \cdot 5^{2001} = 2^{1999} \cdot 5^{1999} \cdot 5^2 = 10^{1999} \cdot 25 = 25\overbrace{0\ldots0}^{1999 \text{ zeros}}.$$

Hence the sum of the digits is 7.

2001 6. **(E)** The last four digits (GHIJ) are either 9753 or 7531, and the remaining odd digit (either 1 or 9) is A, B, or C. Since $A + B + C = 9$, the odd digit among A, B, and C must be 1. Thus the sum of the two even digits in ABC is 8. The three digits in DEF are 864, 642, or 420, leaving the pairs 2 and 0, 8 and 0, or 8 and 6, respectively, as the two even digits in ABC. Of those, only the pair 8 and 0 has sum 8, so ABC is 810, and the required first digit is 8. The only such telephone number is 810-642-9753.

2002A 6. **(E)** When $n = 1$, the inequality becomes $m \leq 1 + m$, which is satisfied by all integers m. Thus, there are infinitely many of the desired values of m.

2008B 6. **Answer (A):** During the year Pete takes

$$44 \times 10^5 + 5 \times 10^4 = 44.5 \times 10^5$$

steps. At 1800 steps per mile, the number of miles Pete walks is

$$\frac{44.5 \times 10^5}{18 \times 10^2} = \frac{44.5}{18} \times 10^3 \approx 2.5 \times 10^3 = 2500.$$

2009A

6. **Answer (E):** Note that

$$12^{mn} = (2^2 \cdot 3)^{mn} = 2^{2mn} \cdot 3^{mn} = (2^m)^{2n} \cdot (3^n)^m = P^{2n}Q^m.$$

Remark: The pair of integers $(2, 1)$ shows that the other choices are not possible.

2010A

6. **Answer (E):** Let $x + 32$ be written in the form $CDDC$. Because x has three digits, $1000 < x + 32 < 1032$, and so $C = 1$ and $D = 0$. Hence $x = 1001 - 32 = 969$, and the sum of the digits of x is $9 + 6 + 9 = 24$.

2012A

6. **Answer (D):** Let the three whole numbers be $a < b < c$. The set of sums of pairs of these numbers is $(a + b, a + c, b + c) = (12, 17, 19)$. Thus $2(a + b + c) = (a + b) + (a + c) + (b + c) = 12 + 17 + 19 = 48$, and $a + b + c = 24$. If follows that $(a, b, c) = (24 - 19, 24 - 17, 24 - 12) = (5, 7, 12)$. Therefore the middle number is 7.

2015B

6. **Answer (A):** There are $13 \cdot 13 = 169$ entries in the body of the table. An entry is odd if and only if both its row factor and its column factor are odd. There are 6 odd whole numbers between 0 and 12, so there are $6 \cdot 6 = 36$ odd entries in the body of the table. The required fraction is $\frac{36}{169} = 0.213\ldots \approx 0.21$.

2004A

7. **(B)** After three rounds the players A, B, and C have 14, 13, and 12 tokens, respectively. Every subsequent three rounds of play reduces each player's supply of tokens by one. After 36 rounds they have 3, 2, and 1 token, respectively, and after the 37^{th} round Player A has no tokens.

2012B

7. **Answer (E):** Consider consecutive red, red, green, green, green lights as a unit. There are $5 \cdot 6 \cdot \frac{1}{12} = 2.5$ feet between corresponding lights in successive units. The 3rd red light begins the 2nd unit, and the 21st red light begins the 11th unit. Therefore the distance between the desired lights is $(11-2) \cdot 2.5 = 22.5$ feet.

2016B

7. **Answer (D):** In the first pass Josh marks out the odd numbers $1, 3, 5, 7, \ldots, 99$, leaving the multiples of 2: $2, 4, 6, 8, \ldots, 100$. In the second pass Josh marks out $2, 6, 10, \ldots, 98$, leaving the multiples of 4: $4, 8, 12, \ldots, 100$. Similarly, in the n^{th} pass Josh marks out the numbers that are not multiples of 2^n, leaving the numbers that are multiples of 2^n. It follows that in the 6^{th} pass Josh marks out the numbers that are multiples of 2^5 but not multiples of 2^6, namely 32 and 92. This leaves 64, the only number in his original list that is a multiple of 2^6. Thus the last number remaining is 64.

2018A

7. **Answer (E):** Because $4000 = 2^5 \cdot 5^3$,

$$4000 \cdot \left(\frac{2}{5}\right)^n = 2^{5+n} \cdot 5^{3-n}.$$

This product will be an integer if and only if both of the factors 2^{5+n} and 5^{3-n} are integers, which happens if and only if both exponents are nonnegative. Therefore the given expression is an integer if and only if $5+n \geq 0$ and $3-n \geq 0$. The solutions are exactly the integers satisfying $-5 \leq n \leq 3$. There are $3-(-5)+1 = 9$ such values.

1999

9. **(D)** The next palindromes after 29792 are 29892, 29992, 30003, and 30103. The difference $30103 - 29792 = 311$ is too far to drive in three hours without exceeding the speed limit of 75 miles per hour. Ashley could have driven $30003 - 29792 = 211$ miles during the three hours for an average speed of $70\frac{1}{3}$ miles per hour.

2006B 9. **(B)** Let the integer have digits a, b, and c, read left to right. Because $1 \le a < b < c$, none of the digits can be zero and c cannot be 2. If $c = 4$, then a and b must each be chosen from the digits 1, 2, and 3. Therefore there are $\binom{3}{2} = 3$ choices for a and b, and for each choice there is one acceptable order. Similarly, for $c = 6$ and $c = 8$ there are, respectively, $\binom{5}{2} = 10$ and $\binom{7}{2} = 21$ choices for a and b. Thus there are altogether $3 + 10 + 21 = 34$ such integers.

2014A 9. **Answer (B):** The five consecutive integers starting with a are a, $a+1$, $a+2$, $a+3$, and $a+4$. Their average is $a+2 = b$. The average of five consecutive integers starting with b is $b+2 = a+4$.

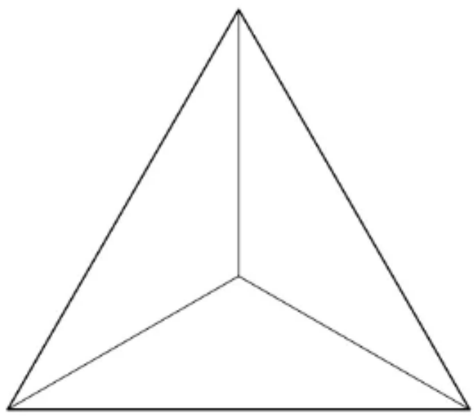
2014A 10. **Answer (B):**

Let h represent the altitude of each of the isosceles triangles from the base on the equilateral triangle. Then the area of each of the congruent isosceles triangles is $\frac{1}{2} \cdot 1 \cdot h = \frac{1}{2}h$. The sum of the areas of the three isosceles triangles is the same as the area of the equilateral triangle, so $3 \cdot \frac{1}{2}h = \frac{1}{4}\sqrt{3}$ and $h = \frac{1}{6}\sqrt{3}$. As a consequence, the Pythagorean Theorem implies that the side length of the isosceles triangles is

$$\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{6}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{12}} = \sqrt{\frac{1}{3}} = \frac{\sqrt{3}}{3}.$$

OR

Suppose that the isosceles triangles are constructed internally with respect to the equilateral triangle. Because the sum of their areas is equal to the area of the equilateral triangle, it follows that the center of the equilateral triangle is a vertex common to all three isosceles triangles. The distance from the center of the equilateral triangle to any of its vertices is two thirds of its height. Thus the required side length is equal to $\frac{2}{3} \cdot \frac{1}{2}\sqrt{3} = \frac{1}{3}\sqrt{3}$.



2014B 10. **Answer (D):** Let m be the total mileage of the trip. Then m must be a multiple of 55. Also, because $m = cba - abc = 99(c - a)$, it is a multiple of 9. Therefore m is a multiple of 495. Because m is at most a 3-digit number and a is not equal to 0, $m = 495$. Therefore $c - a = 5$. Because $a + b + c \leq 7$, the only possible abc is 106, so $a^2 + b^2 + c^2 = 1 + 0 + 36 = 37$.

OR

Let m be the total mileage of the trip. Then m must be a multiple of 55. Also, because $m = cba - abc = 99(c - a)$, $c - a$ is a multiple of 5. Because $a \geq 1$ and $a+b+c \leq 7$, it follows that $c = 6$ and $a = 1$. Therefore $b = 0$, so $a^2+b^2+c^2 = 37$.