UNIT 14 EXERCISES 6-10

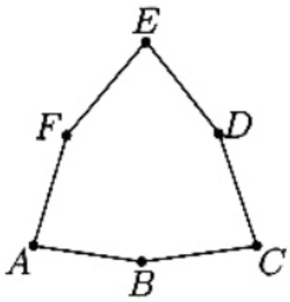
ALGEBRA

2011A

6. **Answer (C):** Let O be the center of the circle, and let the degree measures of the minor and major arcs be $2x$ and $3x$, respectively. Because $2x + 3x = 360°$, it follows that $x = 72°$ and $\angle BOC = 2x = 144°$. In quadrilateral $ABOC$, the segments AB and AC are tangent to the circle, thus $\angle ABO = \angle ACO = 90°$ and $\angle BAC = 360° - (144° + 90° + 90°) = 36°$.

1999

7. **(B)** The sum of the angles in a convex hexagon is $720°$ and each angle must be less than $180°$. If four of the angles are acute, then their sum would be less than $360°$, and therefore at least one of the two remaining angles would be greater than $180°$, a contradiction. Thus there can be at most three acute angles. The hexagon shown has three acute angles, A, C, and E.



OR

The result holds for *any* convex n-gon. The sum of the exterior angles of a convex n-gon is $360°$. Hence at most three of these angles can be obtuse, for otherwise the sum would exceed $360°$. Thus the largest number of acute angles in any convex n-gon is three.

2002B

7. **(B)** Let $n-1$, n, and $n+1$ denote the three integers. Then

$$(n-1)n(n+1) = 8(3n).$$

Since $n \neq 0$, we have $n^2 - 1 = 24$. It follows that $n^2 = 25$ and $n = 5$. Thus,

$$(n-1)^2 + n^2 + (n+1)^2 = 16 + 25 + 36 = 77.$$

2008B 7. **Answer (A):** Note that $(y-x)^2 = (x-y)^2$, so

$$(x-y)^2\$(y-x)^2 = (x-y)^2\$(x-y)^2 = \left((x-y)^2 - (x-y)^2\right)^2 = 0^2 = 0.$$

2014B 7. **Answer (D):** Let $x = \frac{n}{30-n}$ so that $n = \frac{30x}{x+1}$. Because x and $x+1$ are relatively prime, it follows that $x+1$ must be a factor of 30. Because n is positive and less than 30 it follows that $x+1 \geq 2$. Thus $x+1$ equals 2, 3, 5, 6, 10, 15, or 30. Hence there are 7 possible values for n, namely 15, 20, 24, 25, 27, 28, and 29.

2006A 8. **(C)** First note that, in general, the sum of n consecutive integers is n times their median. If the sum is 15, we have the following cases:

if $n = 2$, then the median is 7.5 and the two integers are 7 and 8;

if $n = 3$, then the median is 5 and the three integers are 4, 5, and 6;

if $n = 5$, then the median is 3 and the five integers are 1, 2, 3, 4, and 5.

Because the sum of four consecutive integers is even, 15 cannot be written in such a manner. Also, the sum of more than five consecutive integers must be more than $1+2+3+4+5 = 15$. Hence there are 3 sets satisfying the condition.

Note: It can be shown that the number of sets of two or more consecutive positive integers having a sum of k is equal to the number of odd positive divisors of k, excluding 1.

2005A

8. **(D)** Since A, M, and C are digits we have

$$0 \le A + M + C \le 9 + 9 + 9 = 27.$$

The prime factorization of 2005 is $2005 = 5 \cdot 401$, so

$$100A + 10M + C = 401 \quad \text{and} \quad A + M + C = 5.$$

Hence $A = 4$, $M = 0$, and $C = 1$.

2013A

8. **Answer (D):**

Multiplying the given equation by $xy \neq 0$ yields $x^2y + 2y = xy^2 + 2x$. Thus

$$x^2y - 2x - xy^2 + 2y = x(xy - 2) - y(xy - 2) = (x - y)(xy - 2) = 0.$$

Because $x - y \neq 0$, it follows that $xy = 2$.

2014B

8. **Answer (C):** As indicated by the leftmost column $A + B \le 9$. Then both the second and fourth columns show that $C = 0$. Because A, B, and C are distinct digits, D must be at least 3. The following values for (A, B, C, D) show that D may be any of the 7 digits that are at least 3: $(1, 2, 0, 3)$, $(1, 3, 0, 4)$, $(2, 3, 0, 5)$, $(2, 4, 0, 6)$, $(2, 5, 0, 7)$, $(2, 6, 0, 8)$, $(2, 7, 0, 9)$.

2010B

9. **Answer (E):** Because n is divisible by 20, $n = 2^{2+a} \cdot 5^{1+b} \cdot k$, where a and b are nonnegative integers and k is a positive integer not divisible by 2 or 5. Because $n^2 = 2^{2(2+a)} \cdot 5^{2(1+b)} \cdot k^2$ is a perfect cube, 3 divides $2(2+a)$ and 3 divides $2(1+b)$. Because $n^3 = 2^{3(2+a)} \cdot 5^{3(1+b)} \cdot k^3$ is a perfect square, 2 divides $3(2+a)$ and 2 divides $3(1+b)$. Therefore 6 divides $2+a$ and 6 divides $1+b$. The smallest possible choices for a, b, and k, are $a = 4$, $b = 5$, and $k = 1$. In this case $n = 2^6 \cdot 5^6 = 1{,}000{,}000$, and n has 7 digits.

OR

The only prime factors of 20 are 2 and 5, so n has the form $2^a \cdot 5^b$ for integers $a \geq 2$ and $b \geq 1$. Because n^2 is a perfect cube, $2a$ and $2b$ are both multiples of 3, so a and b are also both multiples of 3. Similarly, because n^3 is a perfect square, a and b are both multiples of 2. Therefore both a and b are multiples of 6. Note that $n = 2^6 \cdot 5^6 = 1{,}000{,}000$ satisfies the given conditions, and n has 7 digits.

2006A

10. **(E)** Suppose that $k = \sqrt{120 - \sqrt{x}}$ is an integer. Then $0 \leq k \leq \sqrt{120}$, and because k is an integer, we have $0 \leq k \leq 10$. Thus there are 11 possible integer values of k. For each such k, the corresponding value of x is $\left(120 - k^2\right)^2$. Because $\left(120 - k^2\right)^2$ is positive and decreasing for $0 \leq k \leq 10$, the 11 values of x are distinct.

2013A

10. **Answer (D):** If n satisfies the equation $\frac{1}{n} = 0.\overline{ab}$, then $\frac{100}{n} = ab.\overline{ab}$ and subtracting gives $\frac{99}{n} = ab$. The positive factors of 99 are 1, 3, 9, 11, 33, and 99. Only $n = 11, 33$, and 99 give a number $\frac{99}{n}$ consisting of two different digits, namely 09, 03, and 01, respectively. Thus the requested sum is $11+33+99 = 143$.

2015A 10. **Answer (E):** Adding 1 to both sides of the equation and factoring yields $(x+1)(y+1) = 81 = 3^4$. Because x and y are distinct positive integers and $x > y$, the only possibility is that $x+1 = 3^3 = 27$ and $y+1 = 3^1 = 3$. Therefore $x = 26$.