UNIT 9 EXERCISES 21-25

SYSTEM OF EQ

2001 21. **(D)** Note that

$$(a+1)(b+1) = ab + a + b + 1 = 524 + 1 = 525 = 3 \cdot 5^2 \cdot 7,$$

and

$$(b+1)(c+1) = bc + b + c + 1 = 146 + 1 = 147 = 3 \cdot 7^2.$$

Since $(a+1)(b+1)$ is a multiple of 25 and $(b+1)(c+1)$ is not a multiple of 5, it follows that $a+1$ must be a multiple of 25. Since $a+1$ divides 525, a is one of 24, 74, 174, or 524. Among these only 24 is a divisor of 8!, so $a = 24$. This implies that $b+1 = 21$, and $b = 20$. From this it follows that $c+1 = 7$ and $c = 6$. Finally, $(c+1)(d+1) = 105 = 3 \cdot 5 \cdot 7$, so $d+1 = 15$ and $d = 14$. Therefore, $a - d = 24 - 14 = 10$.

2012A 21. **Answer (E):** Adding the two equations gives

$$2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ac = 14,$$

so

$$(a-b)^2 + (b-c)^2 + (c-a)^2 = 14.$$

Note that there is a unique way to express 14 as the sum of perfect squares (up to permutations), namely, $14 = 3^2 + 2^2 + 1^2$. Because $a-b$, $b-c$, and $c-a$ are integers with their sum equal to 0 and $a \ge b \ge c$, it follows that $a - c = 3$ and either $a - b = 2$ and $b - c = 1$, or $a - b = 1$ and $b - c = 2$. Therefore either $(a, b, c) = (c+3, c+1, c)$ or $(a, b, c) = (c+3, c+2, c)$. Substituting the relations in the first case into the first given equation yields $2011 = a^2 - c^2 + ab - b^2 = (a-c)(a+c) + (a-b)b = 3(2c+3) + 2(c+1)$. Solving gives $(a, b, c) = (253, 251, 250)$. The second case does not yield an integer solution. Therefore $a = 253$.

2015B 21. **Answer (D):** Assume that there are t steps in this staircase and it took Dash $d+1$ jumps. Then the possible values of t are $5d+1, 5d+2, 5d+3, 5d+4, 5d+5$. On the other hand, it took Cozy $d+20$ jumps, and $t = 2d+39$ or $t = 2d+40$. There are 10 possible combinations but only 3 of them lead to integer values of d: $t = 5d+3 = 2d+39$, or $t = 5d+1 = 2d+40$, or $t = 5d+4 = 2d+40$. The possible values of t are 63, 66, and 64, and $s = 63+66+64 = 193$. The answer is $1+9+3 = 13$.

2005B 23. **(B)** From the given conditions it follows that

$$x+y = 10^z, \quad x^2+y^2 = 10 \cdot 10^z \quad \text{and} \quad 10^{2z} = (x+y)^2 = x^2+2xy+y^2.$$

Thus

$$xy = \frac{1}{2}(10^{2z} - 10 \cdot 10^z).$$

Also

$$(x+y)^3 = 10^{3z} \quad \text{and} \quad x^3+y^3 = (x+y)^3 - 3xy(x+y),$$

which yields

$$\begin{aligned} x^3+y^3 &= 10^{3z} - \frac{3}{2}(10^{2z} - 10 \cdot 10^z)(10^z) \\ &= 10^{3z} - \frac{3}{2}(10^{3z} - 10 \cdot 10^{2z}) = -\frac{1}{2}10^{3z} + 15 \cdot 10^{2z}, \end{aligned}$$

and $a+b = -\frac{1}{2} + 15 = 29/2$.

No other value of $a+b$ is possible for all members of S, because the triple $\left(\frac{1}{2}(1+\sqrt{19}), \frac{1}{2}(1-\sqrt{19}), 0\right)$ is in S, and for this ordered triple, the equation $x^3+y^3 = a \cdot 10^{3z} + b \cdot 10^{2z}$ reduces to $a+b = 29/2$.

2003B 24. (C) Since the system has exactly one solution, the graphs of the two equations must intersect at exactly one point. If $x < a$, the equation $y = |x-a| + |x-b| + |x-c|$ is equivalent to $y = -3x + (a+b+c)$. By similar calculations we obtain

$$y = \begin{cases} -3x + (a+b+c), & \text{if } x < a \\ -x + (-a+b+c), & \text{if } a \le x < b \\ x + (-a-b+c), & \text{if } b \le x < c \\ 3x + (-a-b-c), & \text{if } c \le x. \end{cases}$$

Thus the graph consists of four lines with slopes -3, -1, 1, and 3, and it has corners at $(a, b+c-2a)$, $(b, c-a)$, and $(c, 2c-a-b)$.

On the other hand, the graph of $2x + y = 2003$ is a line whose slope is -2. If the graphs intersect at exactly one point, that point must be $(a, b+c-2a)$. Therefore

$$2003 = 2a + (b+c-2a) = b+c.$$

Since $b < c$, the minimum value of c is 1002.