UNIT 8 EXERCISES 21-25

PROB

2003B 21. (D) Let $\beta = \pi - \alpha$. Apply the Law of Cosines to $\triangle ABC$ to obtain

$$(AC)^2 = 8^2 + 5^2 - 2(8)(5)\cos\beta = 89 - 80\cos\beta.$$

Thus $AC < 7$ if and only if

$$89 - 80\cos\beta < 49, \quad \text{that is, if and only if} \quad \cos\beta > \frac{1}{2}.$$

Therefore we must have $0 < \beta < \frac{\pi}{3}$, and the requested probability is $\frac{\pi/3}{\pi} = \frac{1}{3}$.

2003A

22. (C) Since there are twelve steps between $(0,0)$ and $(5,7)$, A and B can meet only after they have each moved six steps. The possible meeting places are $P_0=(0,6)$, $P_1=(1,5)$, $P_2=(2,4)$, $P_3=(3,3)$, $P_4=(4,2)$, and $P_5=(5,1)$. Let a_i and b_i denote the number of paths to P_i from $(0,0)$ and $(5,7)$, respectively. Since A has to take i steps to the right and B has to take $i+1$ steps down, the number of ways in which A and B can meet at P_i is

$$a_i \cdot b_i = \binom{6}{i}\binom{6}{i+1}.$$

Since A and B can each take 2^6 paths in six steps, the probability that they meet is

$$\sum_{i=0}^{5}\left(\frac{a_i}{2^6}\right)\left(\frac{b_i}{2^6}\right) = \frac{\binom{6}{0}\binom{6}{1}+\binom{6}{1}\binom{6}{2}+\binom{6}{2}\binom{6}{3}+\binom{6}{3}\binom{6}{4}+\binom{6}{4}\binom{6}{5}+\binom{6}{5}\binom{6}{6}}{2^{12}}$$

$$= \frac{99}{512} \approx 0.20.$$

OR

Consider the $\binom{12}{5}$ walks that start at $(0,0)$, end at $(5,7)$, and consist of 12 steps, each one either up or to the right. There is a one-to-one correspondence between these walks and the set of (A,B)-paths where A and B meet. In particular, given one of the $\binom{12}{5}$ walks from $(0,0)$ to $(5,7)$, the path followed by A consists of the the first six steps of the walk, and the path followed by B is obtained by starting at $(5,7)$ and reversing the last six steps of the walk. There are 2^6 paths that take 6 steps from $(0,0)$ and 2^6 paths that take 6 steps from $(5,7)$, so there are 2^{12} pairs of paths that A and B can take. The probability that they meet is

$$P = \frac{1}{2^{12}}\binom{12}{5} = \frac{99}{2^9}.$$

2008B

22. **Answer (E):** The four vacant spaces can be located in any of $\binom{16}{4} = 1820$ combinations of positions. The arrangements in which Auntie Em is unable to park may be divided into two cases. If the rightmost space is occupied, then every vacant space is immediately to the left of an occupied space. Let X denote the union of a vacant space and the occupied space immediately to its right, and let Y denote a single occupied space not immediately to the right of a vacant space. The arrangement of cars and spaces can be represented by a sequence of four X's and eight Y's in some order, and there are $\binom{12}{4} = 495$ possible orders. If the rightmost space is vacant, the arrangement in the remaining 15 spaces can be represented by a sequence of three X's and nine Y's in some order, and there are $\binom{12}{3} = 220$ possible orders. Therefore there are $1820 - 495 - 220 = 1105$ arrangements in which Auntie Em can park, and the requested probability is $\frac{1105}{1820} = \frac{17}{28}$.

OR

Let O denote an occupied space, and let V denote a vacant space. The problem is equivalent to finding the probability p that in a string of 12 O's and 4 V's, there are at least two consecutive V's. Then $1-p$ is the probability that no two V's are consecutive. In a string of 12 O's, there are 13 spaces in which to insert 4 V's to create a string in which no two V's are consecutive. Thus

$$p = 1 - \frac{\binom{13}{4}}{\binom{16}{4}} = \frac{17}{28}.$$

2013A 22. **Answer (E):** Let n be a 6-digit palindrome, $m = \frac{n}{11}$, and suppose m is a palindrome as well. First, if m is a 4-digit number, then $n = 11m < 11 \cdot 10^5 = 10^6 + 10^5$. Thus the first and last digit of n is 1. Thus the last digit of m is 1 and then the first digit of m must be 1 as well. Then $m \leq 1991 < 2000$ and $n = 11m < 11 \cdot 2000 = 22\,000$, which is a contradiction. Therefore m is a 5-digit number $abcba$. If $a + b \leq 9$ and $b + c \leq 9$, then there are no carries in the sum $n = 11m = abcba0 + abcba$; thus the digits of n in order are a, $a+b$, $b+c$, $b+c$, $a+b$, and a. Conversely, if $a + b \geq 10$, then the first digit of n is $a+1$ and the last digit a; and if $a + b \leq 9$ but $b + c \geq 10$, then the second digit of n is $a+b+1$ if $a+b < 9$, or 0 if $a+b = 9$, and the previous to last digit is $a+b$. In any case n is not a palindrome. Therefore $n = 11m$ is a palindrome if and only if $a + b \leq 9$ and $b + c \leq 9$.

Thus the number of pairs (m, n) is equal to

$$\sum_{b=0}^{9} \sum_{c=0}^{9-b} \sum_{a=1}^{9-b} 1 = \sum_{b=0}^{9} (10-b)(9-b).$$

Letting $j = 10 - b$ gives

$$\sum_{j=1}^{10} j(j-1) = \frac{10 \cdot 11 \cdot 21}{6} - \frac{10 \cdot 11}{2} = 330.$$

The total number of 6-digit palindromes $abccba$ is determined by 10 choices for each of b and c, and 9 choices for a, for a total of $9 \cdot 10^2 = 900$. Thus the required probability is $\frac{330}{900} = \frac{11}{30}$.

2014B 22. **Answer (C):** First note that once the frog is on pad 5, it has probability $\frac{1}{2}$ of eventually being eaten by the snake, and a probability $\frac{1}{2}$ of eventually exiting the pond without being eaten. It is therefore necessary only to determine the probability that the frog on pad 1 will reach pad 5 before being eaten.

Consider the frog's jumps in pairs. The frog on pad 1 will advance to pad 3 with probability $\frac{9}{10} \cdot \frac{8}{10} = \frac{72}{100}$, will be back at pad 1 with probability $\frac{9}{10} \cdot \frac{2}{10} = \frac{18}{100}$, and will retreat to pad 0 and be eaten with probability $\frac{1}{10}$. Because the frog will eventually make it to pad 3 or make it to pad 0, the probability that it ultimately makes it to pad 3 is $\frac{72}{100} \div \left(\frac{72}{100} + \frac{10}{100}\right) = \frac{36}{41}$, and the probability that it ultimately makes it to pad 0 is $\frac{10}{100} \div \left(\frac{72}{100} + \frac{10}{100}\right) = \frac{5}{41}$.

Similarly, in a pair of jumps the frog will advance from pad 3 to pad 5 with probability $\frac{7}{10} \cdot \frac{6}{10} = \frac{42}{100}$, will be back at pad 3 with probability $\frac{7}{10} \cdot \frac{4}{10} + \frac{3}{10} \cdot \frac{8}{10} = \frac{52}{100}$, and will retreat to pad 1 with probability $\frac{3}{10} \cdot \frac{2}{10} = \frac{6}{100}$. Because the frog will ultimately make it to pad 5 or pad 1 from pad 3, the probability that it ultimately makes it to pad 5 is $\frac{42}{100} \div \left(\frac{42}{100} + \frac{6}{100}\right) = \frac{7}{8}$, and the probability that it ultimately makes it to pad 1 is $\frac{6}{100} \div \left(\frac{42}{100} + \frac{6}{100}\right) = \frac{1}{8}$.

The sequences of pairs of moves by which the frog will advance to pad 5 without being eaten are

$$1 \to 3 \to 5, 1 \to 3 \to 1 \to 3 \to 5, 1 \to 3 \to 1 \to 3 \to 1 \to 3 \to 5,$$

and so on. The sum of the respective probabilities of reaching pad 5 is then

$$\begin{aligned}
&\frac{36}{41} \cdot \frac{7}{8} + \frac{36}{41} \cdot \frac{1}{8} \cdot \frac{36}{41} \cdot \frac{7}{8} + \frac{36}{41} \cdot \frac{1}{8} \cdot \frac{36}{41} \cdot \frac{1}{8} \cdot \frac{36}{41} \cdot \frac{7}{8} + \cdots \\
&= \frac{63}{82}\left(1 + \frac{9}{82} + \left(\frac{9}{82}\right)^2 + \cdots\right) \\
&= \frac{63}{82} \div \left(1 - \frac{9}{82}\right) \\
&= \frac{63}{73}.
\end{aligned}$$

Therefore the requested probability is $\frac{1}{2} \cdot \frac{63}{73} = \frac{63}{146}$.

OR

For $1 \le j \le 5$, let p_j be the probability that the frog eventually reaches pad 10 starting at pad j. By symmetry $p_5 = \frac{1}{2}$. For the frog to reach pad 10 starting from pad 4, the frog goes either to pad 3 with probability $\frac{2}{5}$ or to pad 5 with probability $\frac{3}{5}$, and then continues on a successful sequence from either of these pads. Thus $p_4 = \frac{2}{5}p_3 + \frac{3}{5}p_5 = \frac{2}{5}p_3 + \frac{3}{10}$. Similarly, to reach pad 10 starting from pad 3, the frog goes either to pad 2 with probability $\frac{3}{10}$ or to pad 4 with probability $\frac{7}{10}$. Thus $p_3 = \frac{3}{10}p_2 + \frac{7}{10}p_4$, and substituting from the previous equation for p_4 gives $p_3 = \frac{5}{12}p_2 + \frac{7}{24}$. In the same way, $p_2 = \frac{1}{5}p_1 + \frac{4}{5}p_3$ and after substituting for p_3 gives $p_2 = \frac{3}{10}p_1 + \frac{7}{20}$. Lastly, for the frog to escape starting from pad 1, it is necessary for it to get to pad 2 with probability $\frac{9}{10}$, and then escape starting from pad 2. Thus $p_1 = \frac{9}{10}p_2 = \frac{9}{10}\left(\frac{3}{10}p_1 + \frac{7}{20}\right)$, and solving the equation gives $p_1 = \frac{63}{146}$.

Note: This type of random process is called a Markov process.

2017B

22. **Answer (B):** There are $4 \cdot 3 = 12$ outcomes for each set of draws and therefore 12^4 outcomes in all. To count the number of outcomes in which each player will end up with four coins, note that this can happen in four ways:

• For some permutation (w, x, y, z) of $\{\text{Abby}, \text{Bernardo}, \text{Carl}, \text{Debra}\}$, the outcomes of the four draws are that w gives a coin to x, x gives a coin to y, y gives a coin to z, and z gives a coin to w, in one of $4! = 24$ orders. There are 3 ways to choose whom Abby gives her coin to and 2 ways to choose whom that person gives his or her coin to, which makes 6 ways to choose the givers and receivers for these transaction. Therefore there are $24 \cdot 6 = 144$ ways for this to happen.

• One pair of the players exchange coins, and the other two players also exchange coins, in one of $4! = 24$ orders. There are 3 ways to choose the pairings. Therefore there are $24 \cdot 3 = 72$ ways for this to happen.

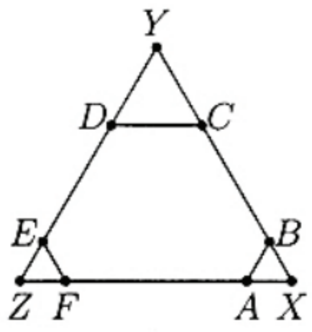
• Two of the players exchange coins twice. There are $\binom{4}{2} = 6$ ways to choose those players and $\binom{4}{2} = 6$ ways to choose the orders of the exchanges, for a total of $6 \cdot 6 = 36$ ways for this to happen.

• One of the players is involved in all four transactions, giving and receiving a coin from each of two others. There are 4 ways to choose this player, 3 ways to choose the other two players, and $4! = 24$ ways to choose the order in which the transactions will take place. Therefore there are $4 \cdot 3 \cdot 24 = 288$ ways for this to happen.

In all, there are $144 + 72 + 36 + 288 = 540$ outcomes that will result in each player having four coins. The requested probability is $\frac{540}{12^4} = \frac{5}{192}$.

1999

23. **(E)** Extend $\overline{FA}$ and $\overline{CB}$ to meet at X, $\overline{BC}$ and $\overline{ED}$ to meet at Y, and $\overline{DE}$ and $\overline{AF}$ to meet at Z. The interior angles of the hexagon are $120°$. Thus the triangles XYZ, ABX, CDY, and EFZ are equilateral. Since $AB = 1$, $BX = 1$. Since $CD = 2$, $CY = 2$. Thus $XY = 7$ and $YZ = 7$. Since $YD = 2$ and $DE = 4, EZ = 1$. The area of the hexagon can be found by subtracting the areas of the three small triangles from the area of the large triangle:



$$7^2\left(\frac{\sqrt{3}}{4}\right) - 1^2\left(\frac{\sqrt{3}}{4}\right) - 2^2\left(\frac{\sqrt{3}}{4}\right) - 1^2\left(\frac{\sqrt{3}}{4}\right) = \frac{43\sqrt{3}}{4}.$$

2005A 23. **(B)** Let $a = 2^j$ and $b = 2^k$. Then

$$\log_a b = \log_{2^j} 2^k = \frac{\log 2^k}{\log 2^j} = \frac{k \log 2}{j \log 2} = \frac{k}{j},$$

so $\log_a b$ is an integer if and only if k is an integer multiple of j. For each j, the number of integer multiples of j that are at most 25 is $\left\lfloor \frac{25}{j} \right\rfloor$. Because $j \neq k$, the number of possible values of k for each j is $\left\lfloor \frac{25}{j} \right\rfloor - 1$. Hence the total number of ordered pairs (a, b) is

$$\sum_{j=1}^{25} \left(\left\lfloor \frac{25}{j} \right\rfloor - 1 \right) = 24 + 11 + 7 + 5 + 4 + 3 + 2(2) + 4(1) = 62.$$

Since the total number of possibilities for a and b is $25 \cdot 24$, the probability that $\log_a b$ is an integer is

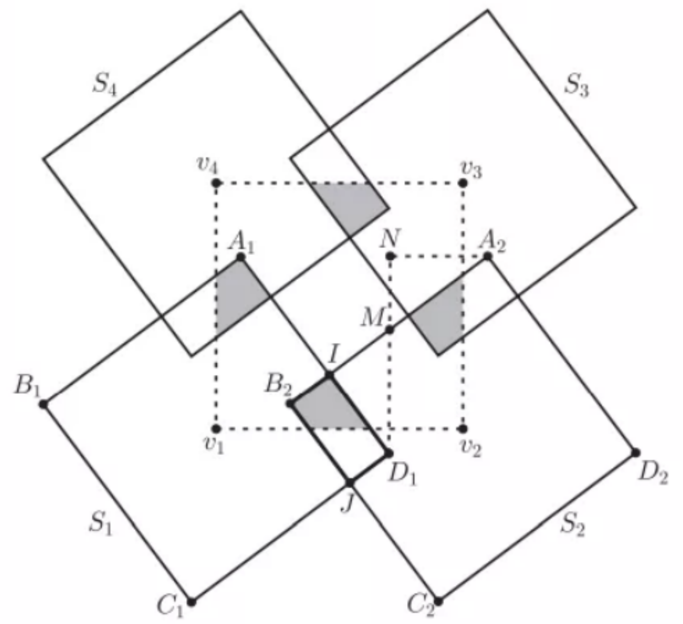
$$\frac{62}{25 \cdot 24} = \frac{31}{300}.$$

2012A

23. **Answer (C):** Consider the unit square U with vertices $v_1 = (0,0), v_2 = (1,0), v_3 = (1,1)$, and $v_4 = (0,1)$, and the squares $S_i = T(v_i)$ with $i = 1,2,3,4$. Note that $T(v)$ contains v_i if and only if $v \in S_i$. First choose a point $v = (x,y)$ uniformly at random over all pairs of real numbers (x,y) such that $0 \le x \le 1$ and $0 \le y \le 1$. In this case, the probability that $T(v)$ contains v_i and v_j is the

area of the intersection of the squares U, S_i, and S_j. This intersection is empty when v_iv_j is a diagonal of U and it is equal to $\text{Area}(U \cap S_i \cap S_j)$ when v_iv_j is a side of U. By symmetry, the probability that $T(v)$ contains two vertices of U is $4 \cdot \text{Area}(U \cap S_1 \cap S_2) = 2 \cdot \text{Area}(S_1 \cap S_2)$. By periodicity, this probability is the same as when the point $v = (x,y)$ is chosen uniformly at random over all pairs of real numbers (x,y) such that $0 \le x \le 2012$ and $0 \le y \le 2012$.

For $i = 1$ and 2, let A_i, B_i, C_i, and D_i be the vertices of S_i in counterclockwise order, where $A_1 = (0.1, 0.7)$ and $A_2 = (1.1, 0.7)$. Then $B_2 = (0.3, 0.1)$ and $D_1 = (0.7, -0.1)$. Let $M = (0.7, 0.4)$ be the midpoint of A_2B_2 and $N = (0.7, 0.7)$. Let $I \in \overline{A_2B_2}$ and $J \in \overline{C_1D_1}$ be the points of intersection of the boundaries of S_1 and S_2. Then $S_1 \cap S_2$ is the rectangle IB_2JD_1. Because D_1, M, and N are collinear and $D_1M = MA_2 = 0.5$, the right triangles A_2NM and D_1IM are congruent. Hence $ID_1 = NA_2 = 1.1 - 0.7 = 0.4$ and $IB_2 = MB_2 - MI = MB_2 - MN = 0.5 - 0.3 = 0.2$. Therefore $\text{Area}(S_1 \cap S_2) = \text{Area}(IB_2JD_1) = 0.2 \cdot 0.4 = 0.08$, and thus the required probability is 0.16.

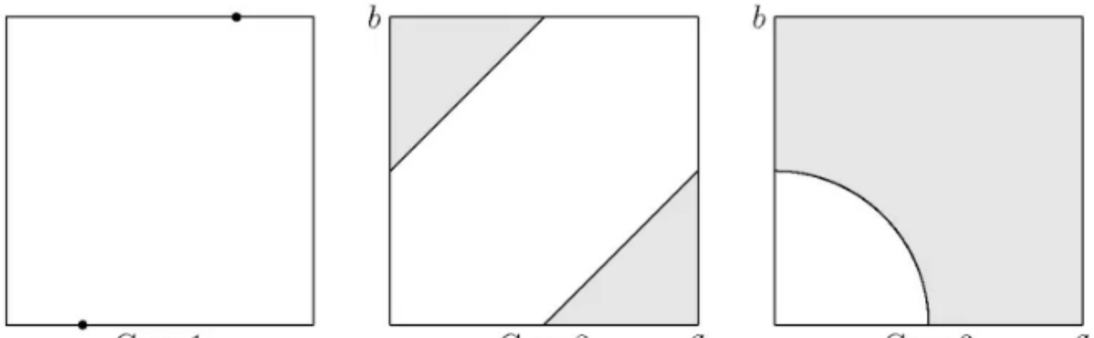


2015A 23. **Answer (A):** Let the square have vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$, and consider three cases.

Case 1: The chosen points are on opposite sides of the square. In this case the distance between the points is at least $\frac{1}{2}$ with probability 1.

Case 2: The chosen points are on the same side of the square. It may be assumed that the points are $(a,0)$ and $(b,0)$. The pairs of points in the ab-plane that meet the requirement are those within the square $0 \le a \le 1$, $0 \le b \le 1$ that satisfy either $b \ge a + \frac{1}{2}$ or $b \le a - \frac{1}{2}$. These inequalities describe the union of two isosceles right triangles with leg length $\frac{1}{2}$, together with their interiors. The area of the region is $\frac{1}{4}$, and the area of the square is 1, so the probability that the pair of points meets the requirement in this case is $\frac{1}{4}$.

Case 3: The chosen points are on adjacent sides of the square. It may be assumed that the points are $(a,0)$ and $(0,b)$. The pairs of points in the ab-plane that meet the requirement are those within the square $0 \le a \le 1$, $0 \le b \le 1$ that satisfy $\sqrt{a^2+b^2} \ge \frac{1}{2}$. These inequalities describe the region inside the square and outside a quarter-circle of radius $\frac{1}{2}$. The area of this region is $1 - \frac{1}{4}\pi(\frac{1}{2})^2 = 1 - \frac{\pi}{16}$, which is also the probability that the pair of points meets the requirement in this case.



Case 1 Case 2 Case 3

Cases 1 and 2 each occur with probability $\frac{1}{4}$, and Case 3 occurs with probability $\frac{1}{2}$. The requested probability is

$$\frac{1}{4} \cdot 1 + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{2}\left(1 - \frac{\pi}{16}\right) = \frac{26-\pi}{32},$$

and $a+b+c = 59$.

2016A 23. **Answer (C):** Let the chosen numbers be x, y, and z. The set of possible ordered triples (x, y, z) forms a solid unit cube, two of whose vertices are $(0, 0, 0)$ and $(1, 1, 1)$. The numbers fail to be the side lengths of a triangle with positive area if and only if one of the numbers is at least as great as the sum of the other two. The ordered triples that satisfy $z \geq x + y$ lie in the region on and above the plane $z = x + y$. The intersection of this region with the solid cube is a solid tetrahedron with vertices $(0, 0, 0)$, $(0, 0, 1)$, $(0, 1, 1)$, and $(1, 0, 1)$. The volume of this tetrahedron is $\frac{1}{6}$. The intersections of the solid cube with the regions defined by the inequalities $y \geq x + z$ and $x \geq y + z$ are solid tetrahedra with the same volume. Because at most one of the inequalities $z > x + y$, $y > x + z$, and $x > y + z$ can be true for any choice of x, y, and z, the three tetrahedra have disjoint interiors. Thus the required probability is $1 - 3 \cdot \frac{1}{6} = \frac{1}{2}$.

OR

As in the first solution, the set of possible ordered triples (x, y, z) forms a solid unit cube. First consider only the points for which $x > y$ and $x > z$. These points form a square pyramid whose vertex is $(0, 0, 0)$ and whose base has vertices at $(0, 0, 1)$, $(1, 0, 1)$, $(1, 1, 1)$, and $(0, 1, 1)$. Such an ordered triple corresponds to the side lengths of a triangle if and only if $z < x + y$. The plane $z = x + y$ passes through the vertex of the pyramid and bisects its base, so it bisects the volume of the pyramid. The probability of forming a triangle is the same as the probability of not forming a triangle. The same argument applies when y or z is the largest element in the triple. The probability of any two of x, y, and z being equal is 0, so this case can be ignored. Thus this event and its complement are equally likely; the probability is $\frac{1}{2}$.

2013A

24. **Answer (E):** Assume without loss of generality that the regular 12-gon is inscribed in a circle of radius 1. Every segment with endpoints in the 12-gon subtends an angle of $\frac{360}{12}k = 30k$ degrees for some $1 \le k \le 6$. Let d_k be the length of those segments that subtend an angle of $30k$ degrees. There are 12 such segments of length d_k for every $1 \le k \le 5$ and 6 segments of length d_6. Because $d_k = 2\sin(15k^\circ)$, it follows that $d_2 = 2\sin(30^\circ) = 1$, $d_3 = 2\sin(45^\circ) = \sqrt{2}$, $d_4 = 2\sin(60^\circ) = \sqrt{3}$, $d_6 = 2\sin(90^\circ) = 2$,

$$\begin{aligned} d_1 &= 2\sin(15^\circ) = 2\sin(45^\circ - 30^\circ) \\ &= 2\sin(45^\circ)\cos(30^\circ) - 2\sin(30^\circ)\cos(45^\circ) = \frac{\sqrt{6}-\sqrt{2}}{2}, \text{ and} \\ d_5 &= 2\sin(75^\circ) = 2\sin(45^\circ + 30^\circ) \\ &= 2\sin(45^\circ)\cos(30^\circ) + 2\sin(30^\circ)\cos(45^\circ) = \frac{\sqrt{6}+\sqrt{2}}{2}. \end{aligned}$$

If $a \le b \le c$, then $d_a \le d_b \le d_c$ and the segments with lengths d_a, d_b, and d_c do not form a triangle with positive area if and only if $d_c \ge d_a + d_b$. Because $d_2 = 1 < \sqrt{6} - \sqrt{2} = 2d_1 < \sqrt{2} = d_3$, it follows that for $(a,b,c) \in \{(1,1,3),(1,1,4),(1,1,5),(1,1,6)\}$, the segments of lengths d_a, d_b, d_c do not form a triangle with positive area. Similarly,

$$\begin{aligned} d_3 &= \sqrt{2} < \frac{\sqrt{6}-\sqrt{2}}{2} + 1 = d_1 + d_2 < \sqrt{3} = d_4, \\ d_4 &< d_5 = \frac{\sqrt{6}+\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{2} + \sqrt{2} = d_1 + d_3, \text{ and} \\ d_5 &< d_6 = 2 = 1 + 1 = 2d_2, \end{aligned}$$

so for $(a,b,c) \in \{(1,2,4),(1,2,5),(1,2,6),(1,3,5),(1,3,6),(2,2,6)\}$, the segments of lengths d_a, d_b, d_c do not form a triangle with positive area. Finally, if $a \ge 2$ and $b \ge 3$, then $d_a + d_b \ge d_2 + d_3 = 1 + \sqrt{2} > 2 \ge d_c$, and also if $a \ge 3$, then $d_a + d_b \ge 2d_3 = 2\sqrt{2} > 2 = d_c$. Therefore the complete list of forbidden triples (d_a, d_b, d_c) is given by $(a,b,c) \in \{(1,1,3), (1,1,4), (1,1,5),(1,1,6),(1,2,4),(1,2,5),(1,2,6),(1,3,5),(1,3,6),(2,2,6)\}$.

For each $(a,b,c) \in \{(1,1,3),(1,1,4),(1,1,5)\}$, there are $\binom{12}{2}$ pairs of segments of length d_a and 12 segments of length d_c. For each $(a,b,c) \in \{(1,1,6),(2,2,6)\}$, there are $\binom{12}{2}$ pairs of segments of length d_a and 6 segments of length d_c. For each $(a,b,c) \in \{(1,2,4),(1,2,5),(1,3,5)\}$, there are 12^3 triples of segments with lengths d_a, d_b, and d_c. Finally, for each $(a,b,c) \in \{(1,2,6),(1,3,6)\}$, there are 12^2 pairs of segments with lengths d_a and d_b, and 6 segments of length d_c. Because the total number of triples of segments equals $\binom{\binom{12}{2}}{3} = \binom{66}{3}$, the required probability equals

$$\begin{aligned} &1 - \frac{3 \cdot 12 \cdot \binom{12}{2} + 2 \cdot 6 \cdot \binom{12}{2} + 3 \cdot 12^3 + 2 \cdot 12^2 \cdot 6}{\binom{66}{3}} \\ &= 1 - \frac{63}{286} = \frac{223}{286}. \end{aligned}$$

2018A

24. **Answer (B):** Because Alice and Bob are choosing their numbers uniformly at random, the cases in which two or three of the chosen numbers are equal have probability 0 and can be ignored. Suppose Carol chooses the number c. She will win if her number is greater than Alice's number and less than Bob's, and she will win if her number is less than Alice's number and greater than Bob's. There are three cases.

- If $c \leq \frac{1}{2}$, then Carol's number is automatically less than Bob's, so her chance of winning is the probability that Alice's number is less than c, which is just c. The best that Carol can do in this case is to choose $c = \frac{1}{2}$, in which case her chance of winning is $\frac{1}{2}$.
- If $c \geq \frac{2}{3}$, then Carol's number is automatically greater than Bob's, so her chance of winning is the probability that Alice's number is greater than c, which is just $1 - c$. The best that Carol can do in this case is to choose $c = \frac{2}{3}$, in which case her chance of winning is $\frac{1}{3}$.
- Finally suppose that $\frac{1}{2} < c < \frac{2}{3}$. The probability that Carol's number is less than Bob's is

$$\frac{\frac{2}{3} - c}{\frac{2}{3} - \frac{1}{2}} = 4 - 6c,$$

so the probability that her number is greater than Alice's and less than Bob's is $c(4 - 6c)$. Similarly, the probability that her number is less than Alice's and greater than Bob's is $(1 - c)(6c - 3)$. Carol's probability of winning in this case is therefore

$$c(4 - 6c) + (1 - c)(6c - 3) = -12c^2 + 13c - 3.$$

The value of a quadratic polynomial with a negative coefficient on its quadratic term is maximized at $\frac{-b}{2a}$, where a is the coefficient on its quadratic term and b is the coefficient on its linear term; here that is when $c = \frac{13}{24}$, which is indeed between $\frac{1}{2}$ and $\frac{2}{3}$. Her probability of winning is then

$$-12 \cdot \left(\frac{13}{24}\right)^2 + 13 \cdot \frac{13}{24} - 3 = \frac{25}{48} > \frac{24}{48} = \frac{1}{2}.$$

Because the probability of winning in the third case exceeds the probabilities obtained in the first two cases, Carol should choose $\frac{13}{24}$.

2005B

25. **(A)** Because each ant can move from its vertex to any of four adjacent vertices, there are 4^6 possible combinations of moves. In the following, consider only those combinations in which no two ants arrive at the same vertex. Label the vertices as A, B, C, A', B' and C', where A', B' and C' are opposite A, B and C, respectively. Let f be the function that maps each ant's starting vertex onto its final vertex. Then neither of $f(A)$ nor $f(A')$ can be either A or A', and similar statements hold for the other pairs of opposite vertices. Thus there are $4 \cdot 3 = 12$ ordered pairs of values for $f(A)$ and $f(A')$. The vertices $f(A)$ and $f(A')$ are opposite each other in four cases and adjacent to each other in eight.

Suppose that $f(A)$ and $f(A')$ are opposite vertices, and, without loss of generality, that $f(A) = B$ and $f(A') = B'$. Then $f(C)$ must be either A or A' and $f(C')$ must be the other. Similarly, $f(B)$ must be either C or C' and $f(B')$ must be the other. Therefore there are $4 \cdot 2 \cdot 2 = 16$ combinations of moves in which $f(A)$ and $f(A')$ are opposite each other.

Suppose now that $f(A)$ and $f(A')$ are adjacent vertices, and, without loss of generality, that $f(A) = B$ and $f(A') = C$. Then one of $f(B)$ and $f(B')$ must be C' and the other cannot be B'. So there are four possible ordered pairs of values for $f(B)$ and $f(B')$. For each of those there are two possible ordered pairs of values for $f(C)$ and $f(C')$. Therefore there are $8 \cdot 4 \cdot 2 = 64$ combinations of moves in which $f(A)$ and $f(A')$ are adjacent to each other.

Hence the probability that no two ants arrive at the same vertex is

$$\frac{16+64}{4^6} = \frac{5 \cdot 2^4}{2^{12}} = \frac{5}{256}.$$

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25. **Answer (D):** Let

$$100 = qk + r, \text{ with } q, r \in \mathbb{Z} \text{ and } |r| \le \frac{k-1}{2}, \text{ and}$$
$$n = q_1 k + r_1, \text{ with } q_1, r_1 \in \mathbb{Z} \text{ and } |r_1| \le \frac{k-1}{2},$$

so that $\left[\frac{100}{k}\right] = q$ and $\left[\frac{n}{k}\right] = q_1$. Note that $\left[\frac{n+mk}{k}\right] = \left[\frac{n}{k}\right] + m$ for every integer m. Thus n satisfies the required identity if and only if $n + mk$ satisfies the identity for all integers m. Thus all members of a residue class mod k either satisfy the required equality or not; moreover, k divides 99! for every $1 \le k \le 99$, so every residue class mod k in the interval $1 \le n \le 99!$ has the same number of elements. Suppose $r \ge 0$. If $r_1 \ge r - \frac{k-1}{2}$, then

$$100 - n = (q - q_1)\,k + (r - r_1)\,,$$

where $0 \le r - r_1 \le \frac{k-1}{2}$. Thus $\left[\frac{100-n}{k}\right] = q - q_1 = \left[\frac{100}{k}\right] - \left[\frac{n}{k}\right]$. Similarly, if $r_1 < r - \frac{k-1}{2}$, then

$$100 - n = (q - q_1 + 1)\,k + (r - r_1 - k)\,,$$

where $-\frac{k-1}{2} \le r - r_1 - k \le -1$. Thus $\left[\frac{100-n}{k}\right] = q - q_1 + 1 > \left[\frac{100}{k}\right] - \left[\frac{n}{k}\right]$. It follows that the only residue classes r_1 that satisfy the identity are those in the interval $r - \frac{k-1}{2} \le r_1 \le \frac{k-1}{2}$. Thus for $r \ge 0$,

$$P(k) = \frac{1}{k}\left(\frac{k-1}{2} + 1 - \left(r - \frac{k-1}{2}\right)\right) = \frac{k-r}{k} = 1 - \frac{|r|}{k}.$$

Similarly, if $r < 0$ then the identity is satisfied only by the residue classes r_1 in the interval $-\frac{k-1}{2} \le r_1 \le r + \frac{k-1}{2}$. Thus for $r < 0$,

$$P(k) = \frac{1}{k}\left(r + \frac{k-1}{2} + 1 - \left(-\frac{k-1}{2}\right)\right) = \frac{k+r}{k} = 1 - \frac{|r|}{k}.$$

To minimize $P(k)$ in the range $1 \le k \le 99$, where k is odd, first suppose that $r = \frac{k-1}{2}$. Note that $P(k) = \frac{1}{2} + \frac{1}{2k}$, $100 = qk + \frac{k-1}{2}$, and so $201 = k\,(2q+1)$.

The minimum of $P(k)$ in this case is achieved by the largest possible k under this restriction. Because $201 = 3 \cdot 67$, it follows that the largest factor k of 201 in the given range is $k = 67$. In this case $P(67) = \frac{1}{2} + \frac{1}{2 \cdot 67} = \frac{34}{67}$. Second, suppose $r = \frac{1-k}{2}$. In this case $P(k) = \frac{1}{2} + \frac{1}{2k}$ and $199 = k(2q-1)$. Because 199 is prime, it follows that $k = 1$ and $P(k) = 1 > \frac{34}{67}$. Finally, if $|r| \le \frac{k-3}{2}$, then

$$\begin{aligned} P(k) = 1 - \frac{|r|}{k} > 1 - \frac{k-3}{2k} &= \frac{1}{2} + \frac{3}{2k} \\ \ge \frac{1}{2} + \frac{3}{2 \cdot 99} > \frac{1}{2} + \frac{1}{2 \cdot 67} &= \frac{34}{67}. \end{aligned}$$

Therefore the minimum value of $P(k)$ in the required range is $\frac{34}{67}$.