UNIT 7 EXERCISES 21-25

STATS

2011B 21. **Answer (D):** Let the arithmetic and geometric means of x and y be $10a + b$ and $10b + a$, respectively. Then

$$\frac{x+y}{2} = 10a + b \Rightarrow (x+y)^2 = 400a^2 + 80ab + 4b^2$$

and

$$\sqrt{xy} = 10b + a \Rightarrow xy = 100b^2 + 20ab + a^2,$$

so

$$(x-y)^2 = (x+y)^2 - 4xy = 396(a^2 - b^2) = 11 \cdot 6^2 \cdot (a+b)(a-b)$$

Because x and y are distinct, a and b are distinct digits, and the last expression is a perfect square if and only if $a + b = 11$ and $a - b$ is a perfect square. The cases $a - b = 1$, 4, and 9 give solutions $(a, b) = (6, 5)$, $(7.5, 3.5)$, and $(10, 1)$, respectively. Because a and b are digits only the first solution is valid. Thus $(x - y)^2 = 11 \cdot 6^2 \cdot 11 = 66^2$ and $|x - y| = 66$. Note that the given conditions are satisfied if $\{x, y\} = \{32, 98\}$.

2017B 21. **Answer (E):** Let S be the sum of Isabella's 7 scores. Then S is a multiple of 7, and

$$658 = 91 + 92 + 93 + \cdots + 97 \leq S \leq 94 + 95 + 96 + \cdots + 100 = 679,$$

so S is one of 658, 665, 672, or 679. Because $S - 95$ is a multiple of 6, it follows that $S = 665$. Thus the sum of Isabella's first 6 scores was $665 - 95 = 570$, which is a multiple of 5, and the sum of her first 5 scores was also a multiple of 5. Therefore her sixth score must have been a multiple of 5. Because her seventh score was 95 and her scores were all different, her sixth score was 100. One possible sequence of scores is $91, 93, 92, 96, 98, 100, 95$.

2017B

24. **Answer (D):** Let F lie on $\overline{AB}$ so that $\overline{DF} \perp \overline{AB}$. Because $BCDF$ is a rectangle, $\angle FCB \cong \angle DBC \cong \angle CAB \cong \angle BCE$, so E lies on $\overline{CF}$ and it is the foot of the altitude to the hypotenuse in $\triangle CBF$. Therefore $\triangle BEF \sim \triangle CBF \cong \triangle BCD \sim \triangle ABC$. Because

$$\overline{DF} \perp \overline{AB}, \quad \overline{FE} \perp \overline{EB}, \quad \text{and} \quad \frac{AB}{DF} = \frac{AB}{BC} = \frac{BE}{FE},$$

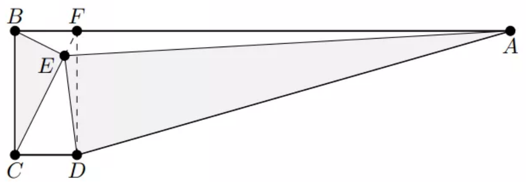
it follows that $\triangle ABE \sim \triangle DFE$. Thus $\angle DEA = \angle DEF - \angle AEF = \angle AEB - \angle AEF = \angle FEB = 90°$. Furthermore,

$$\frac{AE}{ED} = \frac{BE}{EF} = \frac{AB}{BC},$$

so $\triangle AED \sim \triangle ABC$. Assume without loss of generality that $BC = 1$, and let $AB = r > 1$. Because $\frac{AB}{BC} = \frac{BC}{CD}$, it follows that $BF = CD = \frac{1}{r}$. Then

$$17 = \frac{\text{Area}(\triangle AED)}{\text{Area}(\triangle CEB)} = AD^2 = FD^2 + AF^2 = 1 + \left(r - \frac{1}{r}\right)^2,$$

and because $r > 1$ this yields $r^2 - 4r - 1 = 0$, with positive solution $r = 2 + \sqrt{5}$.



OR

Without loss of generality, assume that $BC = 1$. The given conditions imply that the quadrilateral can be placed in the coordinate plane with $C = (0,0)$, $B = (0,1)$, $A = (r,1)$, and $D = (\frac{1}{r}, 0)$. Let E have positive coordinates (x, y). Because $\triangle ABC \sim \triangle CEB$, these coordinates must satisfy

$$\frac{x}{y} = \tan(\angle ECB) = \tan(\angle BAC) = \frac{1}{r}$$

and

$$\sqrt{x^2 + y^2} = \frac{CE}{1} = \frac{r}{\sqrt{1 + r^2}}.$$

Solving this system of equations gives

$$x = \frac{r}{1 + r^2} \quad \text{and} \quad y = \frac{r^2}{1 + r^2}.$$

The area of $\triangle CEB$ is $\frac{x}{2}$. The area of $\triangle AED$ can be computed using the fact that the area of a polygon with vertices (x_1, y_1), (x_2, y_2), ..., (x_n, y_n) in counterclockwise order is

$$\frac{1}{2}\big((x_1y_2 + x_2y_3 + \cdots + x_{n-1}y_n + x_ny_1) - (y_1x_2 + y_2x_3 + \cdots + y_{n-1}x_n + y_nx_1)\big).$$

In this case,

$$\text{Area}(\triangle AED) = \frac{1}{2}\left(y \cdot r + \frac{1}{r} - x - \frac{y}{r}\right).$$

Substituting in the expressions for x and y in terms of r, setting $\text{Area}(\triangle AED) = 17 \cdot \text{Area}(\triangle CEB)$, and simplifying yields the equation $r^4 - 18r^2 + 1 = 0$. Applying the quadratic formula, and noting that $r > 1$, gives $r^2 = 9 + 4\sqrt{5} = (2 + \sqrt{5})^2$, so $r = 2 + \sqrt{5}$.

OR

Let $\theta = \angle ACB$, and without loss of generality assume $BC = 1$. Let F lie on $\overline{AB}$ so that $\overline{DF} \perp \overline{AB}$. Then the requested fraction is $AB = \tan\theta$. Because $\triangle ABC \sim \triangle BCD \sim \triangle CEB \sim \triangle BEF$, it follows that $CD = \cot\theta$, $BE = \cos\theta$, and $CE = \sin\theta$. Then the area of quadrilateral $ABCD$ is $[ABCD] = \frac{1}{2}(\tan\theta + \cot\theta) = \frac{1}{2\sin\theta\cos\theta}$; and the areas of three of the four triangles into which that area can be decomposed are $[ABE] = \frac{1}{2}\tan\theta\cos^2\theta = \frac{1}{2}\sin\theta\cos\theta$, $[BCE] = \frac{1}{2}\sin\theta\cos\theta$, and $[CDE] = \frac{1}{2}\sin^2\theta\cot\theta = \frac{1}{2}\sin\theta\cos\theta$. (Interestingly, the three triangles all have the same area.) Then

$$[AED] = \frac{1}{2\sin\theta\cos\theta} - \frac{3}{2}\sin\theta\cos\theta = 17 \cdot \frac{1}{2}\sin\theta\cos\theta.$$

This last equation simplifies to $20\sin^2\theta\cos^2\theta = 1$, so $(2\sin\theta\cos\theta)^2 = \frac{1}{5}$. Then $\sin(2\theta) = \frac{1}{\sqrt{5}}$, $\cos(2\theta) = \frac{-2}{\sqrt{5}}$ (because $AB > BC$ implies $\frac{\pi}{4} < \theta < \frac{\pi}{2}$), and

$$\tan\theta = \frac{\sin(2\theta)}{\cos(2\theta) + 1} = \frac{1}{-2 + \sqrt{5}} = 2 + \sqrt{5}.$$