UNIT 6 EXERCISES 21-25

COMBINATIONS

2008A 21. **Answer (D):** Call a permutation balanced if $a_1 + a_2 = a_4 + a_5$, and consider the number of balanced permutations. The sum of all five entries is odd, so in a balanced permutation, a_3 must be 1, 3, or 5. For each choice of a_3, there is a unique way to group the remaining four numbers into two sets whose elements have equal sums. For example, if $a_3 = 1$, the two sets must be $\{2, 5\}$ and $\{3, 4\}$. Any one of the four numbers can be a_1, and the value of a_2 is then determined. Either of the two remaining numbers can be a_4, and the value of a_5 is then determined. Thus there are $3 \cdot 2 \cdot 4 = 24$ balanced permutations of $(1, 2, 3, 4, 5)$, and $5! - 24 = 96$ permutations that are not balanced. Call a permutation heavy-headed if $a_1 + a_2 > a_4 + a_5$. Reversing the entries in a heavy-headed permutation yields a unique heavy-tailed permutation, and vice versa, so there are exactly as many heavy-headed permutations as heavy-tailed ones. Therefore the number of heavy-tailed permutations is $\frac{1}{2} \cdot 96 = 48$.

2009B 21. **Answer (A):** Let S_n denote the number of ways that n women in n seats can be reseated so that each woman reseats herself in the seat she occupied before or a seat next to it. It is easy to see that $S_1 = 1$ and $S_2 = 2$. Now consider the case with $n \geq 3$ women, and focus on the woman at the right end of the line. If this woman sits again in this end seat, then the remaining $n - 1$ women can reseat themselves in S_{n-1} ways. If this end woman sits in the seat next to hers, then the former occupant of this new seat must sit on the end. Then the remaining $n - 2$ women can seat themselves in S_{n-2} ways. Thus for $n \geq 3$, $S_n = S_{n-1} + S_{n-2}$. Therefore $(S_1, S_2, \ldots, S_{10}) = (1, 2, 3, 5, 8, 13, 21, 34, 55, 89)$, which are some of the first few terms of the Fibonacci Sequence. Thus $S_{10} = 89$.

2004B 22. **(C)** All the unknown entries can be expressed in terms of b. Since $100e = beh = ceg = def$, it follows that $h = 100/b$, $g = 100/c$, and $f = 100/d$. Comparing rows 1 and 3 then gives

$$50bc = 2\frac{100}{b}\frac{100}{c},$$

from which $c = 20/b$.

Comparing columns 1 and 3 gives

$$50d\frac{100}{c} = 2c\frac{100}{d},$$

from which $d = c/5 = 4/b$.

Finally, $f = 25b$, $g = 5b$, and $e = 10$. All the entries are positive integers if and only if $b = 1$, 2, or 4. The corresponding values for g are 5, 10, and 20, and their sum is 35.

2009B

22. **Answer (C):** Let $B = (b, b)$ and $D = (d, kd)$, so $C = (b + d, b + kd)$. Let $E = (b + d, 0)$ and $F = (0, b + kd)$. Rectangle $AECF$ is the disjoint union of parallelogram $ABCD$, two rectangles with length d and height b, two isosceles right triangles with leg length b, and two right triangles with leg lengths d and kd. It follows that the area of $ABCD$ is

$$(b + d)(b + kd) - 2bd - b^2 - kd^2 = (k - 1)bd.$$

Therefore each parallelogram with the required properties determines, and is determined by, an ordered triple $(k - 1, b, d)$ of positive integers whose product is $1{,}000{,}000 = 2^6 5^6$. The number of ways to distribute the six factors of 2 among the three integers $k-1$, b, and d is $\binom{6+3-1}{3-1} = \binom{8}{2} = 28$. The six factors of 5 can also be distributed in 28 ways, so there are $28^2 = 784$ parallelograms with the required property.

OR

The area of a triangle with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) is

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

Thus the area of $\triangle ABD$ is $\frac{1}{2}(k-1)bd$ and the area of $\triangle CBD$ is the same. Then proceed as in the first solution.

2011A

22. **Answer (D):** Let $T_n = \triangle ABC$. Suppose $a = BC$, $b = AC$, and $c = AB$. Because $\overline{BD}$ and $\overline{BE}$ are both tangent to the incircle of $\triangle ABC$, it follows that $BD = BE$. Similarly, $AD = AF$ and $CE = CF$. Then

$$\begin{aligned} 2BE &= BE + BD = BE + CE + BD + AD - (AF + CF) \\ &= a + c - b, \end{aligned}$$

that is, $BE = \frac{1}{2}(a + c - b)$. Similarly $AD = \frac{1}{2}(b + c - a)$ and $CF = \frac{1}{2}(a + b - c)$. In the given $\triangle ABC$, suppose that $AB = x + 1$, $BC = x - 1$, and $AC = x$. Using the formulas for BE, AD, and CF derived before, it must be true that

$$\begin{aligned} BE &= \frac{1}{2}\left((x-1) + (x+1) - x\right) = \frac{1}{2}x, \\ AD &= \frac{1}{2}\left(x + (x+1) - (x-1)\right) = \frac{1}{2}x + 1, \text{ and} \\ CF &= \frac{1}{2}\left((x-1) + x - (x+1)\right) = \frac{1}{2}x - 1. \end{aligned}$$

Hence both (BC, CA, AB) and (CF, BE, AD) are of the form $(y - 1, y, y + 1)$. This is independent of the values of a, b, and c, so it holds for all T_n. Furthermore, adding the formulas for BE, AD, and CF shows that the perimeter of T_{n+1} equals $\frac{1}{2}(a + b + c)$, and consequently the perimeter of the last triangle T_N in the sequence is

$$\frac{1}{2^{N-1}}(2011 + 2012 + 2013) = \frac{1509}{2^{N-3}}.$$

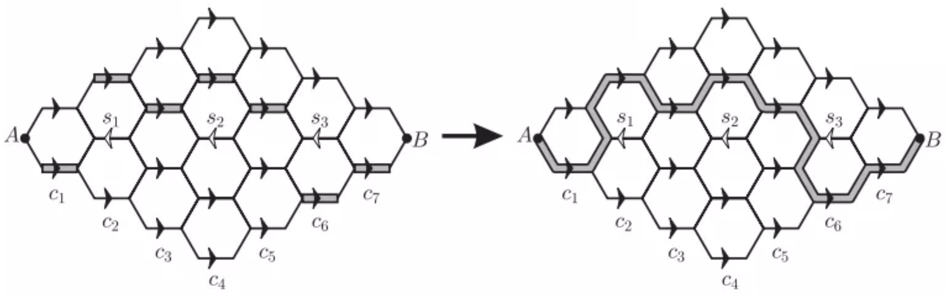
The last member T_N of the sequence will fail to define a successor if for the first time the new lengths fail the Triangle Inequality, that is, if

$$-1 + \frac{2012}{2^N} + \frac{2012}{2^N} \leq 1 + \frac{2012}{2^N}.$$

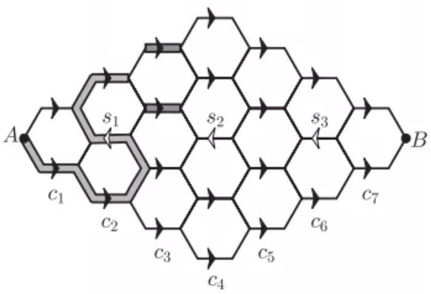
Equivalently, $2012 \leq 2^{N+1}$ which happens for the first time when $N = 10$. Thus the required perimeter of T_N is $\frac{1509}{2^7} = \frac{1509}{128}$.

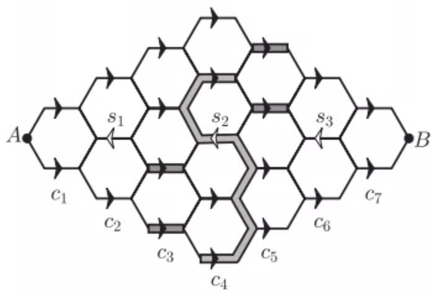
2012B 22. **Answer (E):** Label the columns having arrows as $c_1, c_2, c_3, \ldots, c_7$ according to the figure. Call those segments that can be traveled only from left to right *forward segments*. Call the segments s_1, s_2, and s_3, in columns c_2, c_4, and c_6, respectively, which can be traveled only from right to left, *back segments*. Denote S as the set of back segments traveled for a path.

First suppose that $S = \emptyset$. Because it is not possible to travel a segment more than once, it follows that the path is uniquely determined by choosing one forward segment in each of the columns c_j. There are 2, 2, 4, 4, 4, 2, and 2 choices for the forward segment in columns c_1, c_2, c_3, c_4, c_5, c_6, and c_7, respectively. This gives a total of 2^{10} total paths in this case.



Next suppose that $S = \{s_1\}$. The two forward segments in c_2, together with s_1, need to be part of the path, and once the forward segment from c_1 is chosen, the order in which the segments of c_2 are traveled is determined. Moreover, there are only 2 choices for possible segments in c_3 depending on the last segment traveled in c_2, either the bottom 2 or the top 2. For the rest of the columns, the path is determined by choosing any forward segment. Thus the total number of paths in this case is $2 \cdot 1 \cdot 2 \cdot 4 \cdot 4 \cdot 2 \cdot 2 = 2^8$, and by symmetry this is also the total for the number of paths when $S = \{s_3\}$. A similar argument gives $2 \cdot 1 \cdot 2 \cdot 4 \cdot 2 \cdot 1 \cdot 2 = 2^6$ trips for the case when $S = \{s_1, s_3\}$.





Suppose $S = \{s_2\}$. Because s_2 is traveled, it follows that 2 forward segments in c_4 need to belong to the path, one of them above s_2 (2 choices) and the other below it (2 choices). Once these are determined, there are 2 possible choices for the order in which these segments are traveled: the bottom forward segment first, then s_2, then the top forward segment, or vice versa. Next, there are only 2 possible forward segments that can be selected in c_3 and also only 2 possible forward segments that can be selected in c_5. The forward segments in c_1, c_2, c_6, and c_7 can be freely selected (2 choices each). This gives a total of $(2^3 \cdot 2 \cdot 2) \cdot 2^4 = 2^9$ paths.

If $S = \{s_1, s_2\}$, then the analysis is similar, except for the last step, where the forward segments of c_1 and c_2 are determined by the previous choices. Thus there are $(2^3 \cdot 2 \cdot 2) \cdot 2^2 = 2^7$ possibilities, and by symmetry the same number when $S = \{s_2, s_3\}$.

Finally, if $S = \{s_1, s_2, s_3\}$, then in the last step, all forward segments of c_1, c_2, c_6, and c_7 are determined by the previous choices and hence there are $2^3 \cdot 2 \cdot 2 = 2^5$ possible paths. Altogether the total number of paths is $2^{10} + 2 \cdot 2^8 + 2^6 + 2^9 + 2 \cdot 2^7 + 2^5 = 2400$.

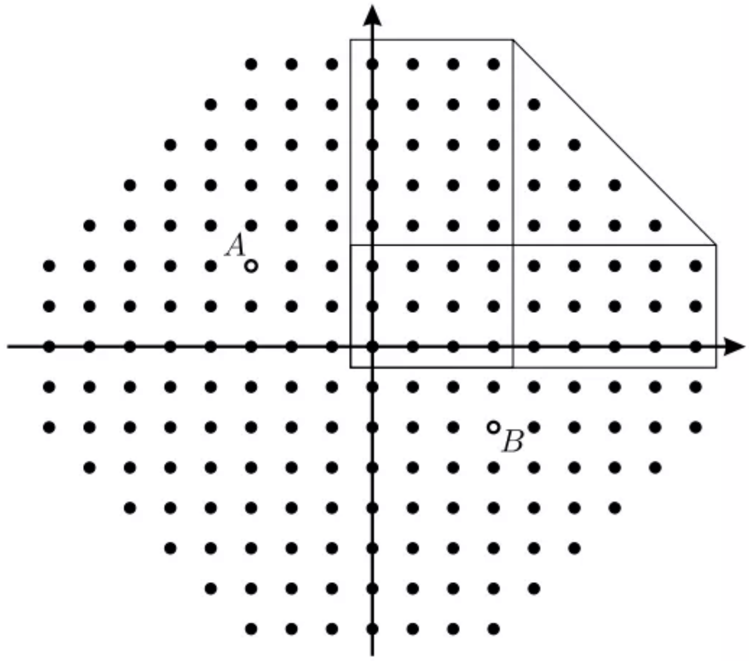
2015B 22. **Answer (D):** To make the analysis easier, suppose first that everyone gets up and moves to the chair directly across the table. The reseating rule now is that each person must sit in the same chair or in an adjacent chair. There must be either 0, 2, 4, or 6 people who choose the same chair; otherwise there would be an odd-sized gap, which would not permit all the people in that gap to sit in an adjacent chair. If no people choose the same chair, then either everyone moves left, which can be done in 1 way, or everyone moves right, which can be done in 1 way, or people swap with a neighbor, which can be done in 2 ways, for a total of 4 possibilities. If two people choose the same chair, then they must be either directly opposite each other or next to each other; there are $3 + 6 = 9$ such pairs. The remaining four people must swap in pairs, and that can be done in just 1 way in each case. If four people choose the same chair, there are 6 ways to choose those people and the other two people swap. Finally, there is 1 way for everyone to choose the same chair. Therefore there are $4 + 9 + 6 + 1 = 20$ ways in which the reseating can be done.

2011B

23. **Answer (C):** Let $X = (x, y)$. The distance traveled by the bug from A to X is at least $|x+3| + |y-2|$. Similarly, the distance traveled by the bug from X to B is at least $|x-3| + |y+2|$. It follows that X belongs to a path from A to B traveled by the bug if and only if

$$d = |x-3| + |x+3| + |y-2| + |y+2| \leq 20.$$

The expression for d is invariant if x is replaced by $-x$ or y is replaced by $-y$. By symmetry, it is enough to count the number of points X with $x \geq 0$ and $y \geq 0$, multiply by 4, and subtract the points that were overcounted, that is those in the x-axis or in the y-axis. Consider four cases:



Case 1. $0 \leq x \leq 3$ and $0 \leq y \leq 2$. In this case $|x-3| + |x+3| = 6$ and $|y-2| + |y+2| = 4$. Thus $d = 10 < 20$ and there are $4 \cdot 3 = 12$ points X in this case. This includes the origin and 5 other points for which $xy = 0$.

Case 2. $0 \leq x \leq 3$ and $y \geq 3$. In this case $|x-3|+|x+3| = 6$ and $|y-2|+|y+2| = 2y$. Thus $d = 6 + 2y \leq 20$ if and only if $y \leq 7$. There are $4 \cdot 5 = 20$ points X in this case. This includes 5 points for which $xy = 0$.

Case 3. $x \geq 4$ and $0 \leq y \leq 2$. In this case $|x-3| + |x+3| = 2x$ and $|y-2| + |y+2| = 4$. Thus $d = 4 + 2x \leq 20$ if and only if $x \leq 8$. There are $5 \cdot 3 = 15$ points X in this case. This includes 5 points for which $xy = 0$.

Case 4. $x \geq 4$ and $y \geq 3$. In this case $|x-3|+|x+3| = 2x$ and $|y-2|+|y+2| = 2y$. Thus $d = 2x + 2y \leq 20$ if and only if $x + y \leq 10$. The number of points X in this case is equal to

$$\sum_{x=4}^{7} \sum_{y=3}^{10-x} 1 = \sum_{x=4}^{7} (10 - x - 2) = \sum_{x=4}^{7} (8 - x) = 4 + 3 + 2 + 1 = 10,$$

and there are no points with $xy = 0$.

By symmetry the required total is $4(12 + 20 + 15 + 10) - 2(5 + 5 + 5) - 3 = 4 \cdot 57 - 2 \cdot 15 - 3 = 195$.

2005A

25. **(C)** Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, and $C(x_3, y_3, z_3)$ be the vertices of such a triangle. Let

$$(\Delta x_k, \Delta y_k, \Delta z_k) = (x_{k+1} - x_k, y_{k+1} - y_k, z_{k+1} - z_k), \text{ for } 1 \le k \le 3,$$

where $(x_4, y_4, z_4) = (x_1, y_1, z_1)$. Then $(|\Delta x_k|, |\Delta y_k|, |\Delta z_k|)$ is a permutation of one of the ordered triples $(0,0,1)$, $(0,0,2)$, $(0,1,1)$, $(0,1,2)$, $(0,2,2)$, $(1,1,1)$, $(1,1,2)$, $(1,2,2)$, or $(2,2,2)$. Since $\triangle ABC$ is equilateral, $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ correspond to permutations of the same ordered triple (a,b,c). Because

$$\sum_{k=1}^{3} \Delta x_k = \sum_{k=1}^{3} \Delta y_k = \sum_{k=1}^{3} \Delta z_k = 0,$$

the sums

$$\sum_{k=1}^{3} |\Delta x_k|, \quad \sum_{k=1}^{3} |\Delta y_k|, \quad \text{and} \quad \sum_{k=1}^{3} |\Delta z_k|$$

are all even. Therefore $(|\Delta x_k|, |\Delta y_k|, |\Delta z_k|)$ is a permutation of one of the triples $(0,0,2)$, $(0,1,1)$, $(0,2,2)$, $(1,1,2)$, or $(2,2,2)$.

If $(a,b,c) = (0,0,2)$, each side of $\triangle ABC$ is parallel to one of the coordinate axes, which is impossible.

If $(a,b,c) = (2,2,2)$, each side of $\triangle ABC$ is an interior diagonal of the $2 \times 2 \times 2$ cube that contains S, which is also impossible.

If $(a,b,c) = (0,2,2)$, each side of $\triangle ABC$ is a face diagonal of the $2 \times 2 \times 2$ cube that contains S. The three faces that join at any vertex determine such a triangle, so the triple $(0,2,2)$ produces a total of 8 triangles.

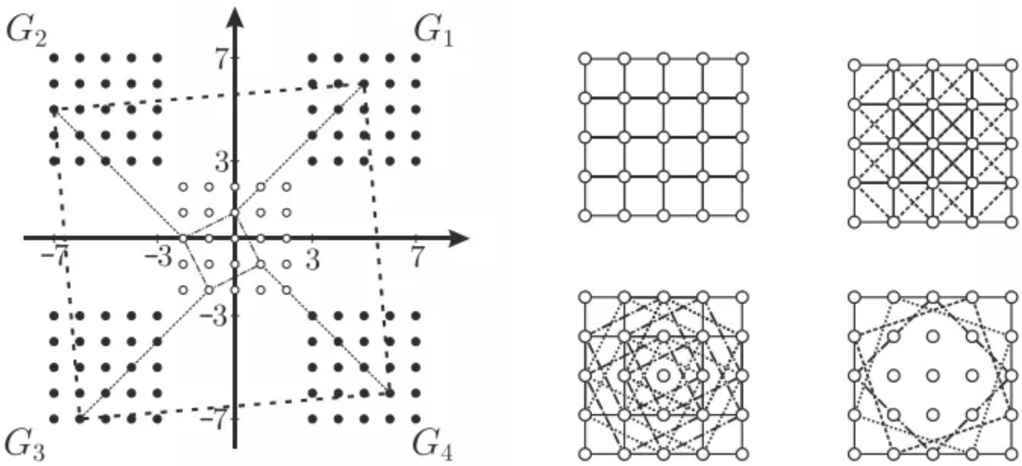
If $(a,b,c) = (0,1,1)$, each side of $\triangle ABC$ is a face diagonal of a unit cube within the larger cube that contains S. There are 8 such unit cubes producing a total of $8 \cdot 8 = 64$ triangles.

There are two types of line segments for which $(a,b,c) = (1,1,2)$. One type joins the center of the face of the $2 \times 2 \times 2$ cube to a vertex on the opposite face. The other type joins the midpoint of one edge of the cube to the midpoint of another edge. Only the second type of segment can be a side of $\triangle ABC$. The midpoint of each of the 12 edges is a vertex of two suitable triangles, so there are $12 \cdot 2/3 = 8$ such triangles.

The total number of triangles is $8 + 64 + 8 = 80$.

2009B 25. **Answer (E):** Let G_i be the subset of G contained in the ith quadrant, $1 \le i \le 4$. For a fixed i, the maximum distance among points in G_i is $4\sqrt{2} < 6$, also the distance from a point in G_i to a point in $G_j \neq G_i$ is at least 6. Thus the required squares are exactly the squares in G with exactly one vertex in each of the G_i. Let $S = p_1p_2p_3p_4$ be a square with vertices $p_i \in G_i$. Let $p_1' = p_1 + (-5,-5)$,

$p_2' = p_2 + (5,-5)$, $p_3' = p_3 + (5,5)$, and $p_4' = p_4 + (-5,5)$. Observe that p_1', p_2', p_3', and p_4' are all lattice points inside the square region G' defined by the points (x,y) with $|x|, |y| \le 2$; moreover, by symmetry, $S' = p_1'p_2'p_3'p_4'$ is either a square or $p_1' = p_2' = p_3' = p_4'$. Reciprocally, if $S' = p_1'p_2'p_3'p_4'$ is a square in G', then the points $p_1 = p_1' + (5,5)$, $p_2 = p_2' + (-5,5)$, $p_3 = p_3' + (-5,-5)$, and $p_4 = p_4' + (5,-5)$ satisfy that $p_i \in G_i$ and $S = p_1p_2p_3p_4$ is a square. The same conclusion holds if $p_1' = p_2' = p_3' = p_4'$. Therefore the required count consists of the number of points in G' plus four times the number of squares with vertices in G'.



There are 5^2 points in G' and the following number of squares with vertices in G': 4^2 of side 1, 3^2 of side 2, 3^2 of side $\sqrt{2}$ (each inscribed in a unique square of side 2), 2^2 of side 3, $2 \cdot 2^2$ of side $\sqrt{5}$ (exactly two inscribed in every square of side 3), 1^2 of side 4, 1^2 of side $2\sqrt{2}$, and $2 \cdot 1^2$ of side $\sqrt{10}$ (exactly two inscribed in the square of side 4). Thus the answer is

$$5^2 + 4 \cdot (4^2 + 2 \cdot 3^2 + 3 \cdot 2^2 + 4 \cdot 1^2) = 25 + 4 \cdot 50 = 225.$$