UNIT 5 EXERCISES 21-25

GEO WORD

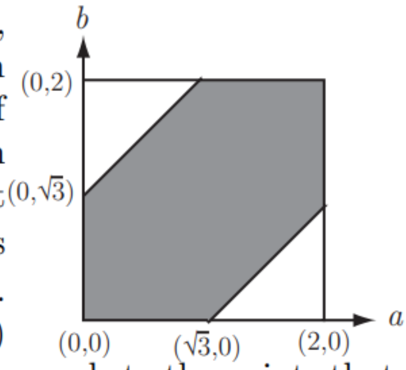
2016B

23. **Answer (A):** In the first octant, the first inequality reduces to $x + y + z \leq 1$, and the inequality defines the region under a plane that intersects the coordinate

axes at $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. By symmetry, the first inequality defines the region inside a regular octahedron centered at the origin and having internal diagonals of length 2. The upper half of this octahedron is a pyramid with altitude 1 and a square base of side length $\sqrt{2}$, so the volume of the octahedron is $2 \cdot \frac{1}{3} \cdot \left(\sqrt{2}\right)^2 \cdot 1 = \frac{4}{3}$. The second inequality defines the region obtained by translating the first region up 1 unit. The intersection of the two regions is bounded by another regular octahedron with internal diagonals of length 1. Because the linear dimensions of the third octahedron are half those of the first, its volume is $\frac{1}{8}$ that of the first, or $\frac{1}{6}$.

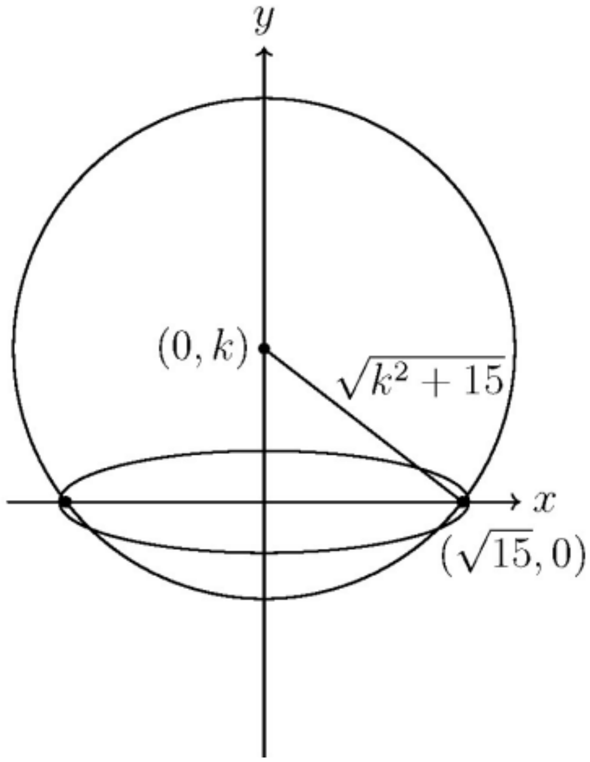
2008B

21. **Answer (E):** Circles A and B both have radius 1, so they intersect if and only if the distance between their centers is no greater than 2. Let the centers of the circles be $(a, 0)$ and $(b, 1)$. The distance between these points is $\sqrt{(a-b)^2+1}$, so the circles intersect if and only if $\sqrt{(a-b)^2+1} \leq 2$. This condition is equivalent to $(a-b)^2 \leq 3$, or $-\sqrt{3} \leq a-b \leq \sqrt{3}$. Points in the square correspond to ordered pairs (a, b) with $0 \leq a \leq 2$ and $0 \leq b \leq 2$. The shaded region corresponds to the points that satisfy $-\sqrt{3} \leq a-b \leq \sqrt{3}$. Its area is $4-\left(2-\sqrt{3}\right)^2$. The requested probability is the area of the shaded region divided by the area of the square, which is



$$\frac{4-\left(2-\sqrt{3}\right)^2}{4}=\frac{4\sqrt{3}-3}{4}.$$

2015A 21. **Answer (D):** The ellipse with equation $x^2 + 16y^2 = 16$ is centered at the origin, with a major axis of length 8 and a minor axis of length 2. If the foci have coordinates $(\pm c, 0)$, then $c^2 + 1^2 = 4^2$. Thus $c = \pm\sqrt{15}$. Any circle passing through both foci must have its center on the y-axis; thus r is at least as large as the distance from the foci to the y-axis. That is, $r \geq \sqrt{15}$. For any $k \geq 0$, the circle of radius $\sqrt{k^2 + 15}$ and center $(0, k)$ passes through both foci (in the interior of the ellipse) and the points $(0, k \pm \sqrt{k^2 + 15})$. The point $(0, k + \sqrt{k^2 + 15})$ is in the exterior of the ellipse since $k + \sqrt{k^2 + 15} > \sqrt{15} > 1$. The point $(0, k - \sqrt{k^2 + 15})$ is in the exterior of the ellipse if and only if $k - \sqrt{k^2 + 15} < -1$, that is, if and only if $k < 7$. Thus, for $k \geq 0$, the circle with center $(0, k)$ intersects the ellipse in four points if and only if $0 \leq k < 7$. As k increases, the radius $r = \sqrt{k^2 + 15}$ increases as well, so the set of possible radii is the interval $[\sqrt{15}, \sqrt{7^2 + 15}) = [\sqrt{15}, 8)$. The requested answer is $\sqrt{15} + 8$.



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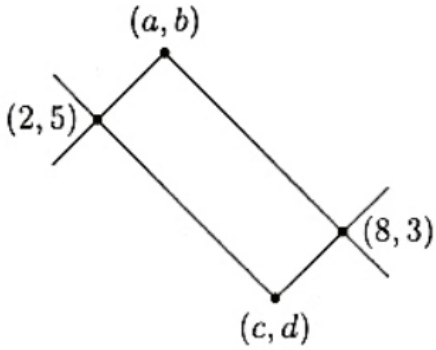

22. **(C)** The first graph is an inverted 'V-shaped' right angle with vertex at (a, b) and the second is a V-shaped right angle with vertex at (c, d). Thus $(a, b), (2, 5), (c, d)$, and $(8, 3)$ are consecutive vertices of a rectangle. The diagonals of this rectangle meet at their common midpoint, so the x-coordinate of this midpoint is $(2+8)/2 = (a+c)/2$. Thus $a+c = 10$.

OR

Use the given information to obtain the equations $5 = -|2-a| + b$, $5 = |2-c| + d$, $3 = -|8-a| + b$, and $3 = |8-c| + d$. Subtract the third from the first to eliminate b and subtract the fourth from the second to eliminate d. The two resulting equations $|8-a| - |2-a| = 2$ and $|2-c| - |8-c| = 2$ can be solved for a and c. To solve the former, first consider all $a \le 2$, for which the equation reduces to $8 - a - (2 - a) = 2$, which has no solutions. Then consider all a in the interval $2 \le a \le 8$, for which the equation reduces to $8 - a - (a - 2) = 2$, which yields $a = 4$. Finally, consider all $a \ge 8$, for which the equation reduces to $a - 8 - (a - 2) = 2$, which has no solutions. The other equation can be solved similarly to show that $c = 6$. Thus $a + c = 10$.

2017A

22. **Answer (E):** Let $A = \{(1,0),(0,1),(-1,0),(0,-1)\}$, let $C = \{(0,0)\}$, and let $I = \{(1,1),(-1,1),(-1,-1),(1,-1)\}$. A particle in A will move to A with probability $\frac{2}{8}$, to C with probability $\frac{1}{8}$, to I with probability $\frac{2}{8}$, and to an interior point of a side of the square with probability $\frac{3}{8}$. Similarly, a particle in C will move to A with probability $\frac{4}{8}$ and to I with probability $\frac{4}{8}$; and a particle in I will move to A with probability $\frac{2}{8}$, to C with probability $\frac{1}{8}$, to a corner of the square with probability $\frac{1}{8}$, and to an interior point of a side of the square with probability $\frac{4}{8}$. Let a, c, and i be the probabilities that the particle will first hit the square at a corner, given that it is currently in A, C, and I, respectively. The transition probabilities noted above lead to the following system of equations.

$$a = \frac{2}{8}a + \frac{1}{8}c + \frac{2}{8}i$$
$$c = \frac{4}{8}a + \frac{4}{8}i$$
$$i = \frac{2}{8}a + \frac{1}{8}c + \frac{1}{8}$$

This system can be solved by elimination to yield $a = \frac{1}{14}$, $c = \frac{4}{35}$, and $i = \frac{11}{70}$. The required fraction is c, whose numerator and denominator sum to 39.

2005B 24. **(A)** Suppose that the triangle has vertices $A(a, a^2)$, $B(b, b^2)$ and $C(c, c^2)$. The slope of line segment $\overline{AB}$ is

$$\frac{b^2 - a^2}{b - a} = b + a,$$

so the slopes of the three sides of the triangle have a sum

$$(b + a) + (c + b) + (a + c) = 2 \cdot \frac{m}{n}.$$

The slope of one side is $2 = \tan\theta$, for some angle θ, and the two remaining sides have slopes

$$\tan\left(\theta \pm \frac{\pi}{3}\right) = \frac{\tan\theta \pm \tan(\pi/3)}{1 \mp \tan\theta\tan(\pi/3)} = \frac{2 \pm \sqrt{3}}{1 \mp 2\sqrt{3}} = -\frac{8 \pm 5\sqrt{3}}{11}.$$

Therefore

$$\frac{m}{n} = \frac{1}{2}\left(2 - \frac{8 + 5\sqrt{3}}{11} - \frac{8 - 5\sqrt{3}}{11}\right) = \frac{3}{11},$$

and $m + n = 14$.

Such a triangle exists. The x-coordinates of its vertices are $(11 \pm 5\sqrt{3})/11$ and $-19/11$.

OR

Define the vertices as in the first solution, with the added stipulations that $a < b$ and $\overline{AB}$ has slope 2. Then

$$2 = \frac{b^2 - a^2}{b - a} = b + a, \quad \text{so} \quad a = 1 - k \text{ and } b = 1 + k,$$

for some $k > 0$. If D is the midpoint of $\overline{AB}$, then

$$D = \left(1, \frac{(1 - k)^2 + (1 + k)^2}{2}\right) = (1, 1 + k^2).$$

The slope of the altitude $\overline{CD}$ is $-1/2$, so

$$1 - c = 2(c^2 - 1 - k^2).$$

Therefore

$$CD^2 = (1 - c)^2 + \left(c^2 - 1 - k^2\right)^2 = \frac{5}{4}(1 - c)^2.$$

Because $\triangle ABC$ is equilateral, we also have

$$CD^2 = \frac{3}{4}AB^2 = \frac{3}{4}\left((2k)^2 + (4k)^2\right) = 15k^2.$$

Hence

$$\frac{5}{4}(1 - c)^2 = 15k^2, \quad \text{so} \quad k^2 = \frac{(1 - c)^2}{12}.$$

Substitution into the equation $1 - c = 2(c^2 - 1 - k^2)$ yields $c = 1$ or $c = -19/11$. Because $c < 1$, it follows that

$$a + b + c = 2 - \frac{19}{11} = \frac{3}{11} = \frac{m}{n}, \quad \text{so} \quad m + n = 14.$$