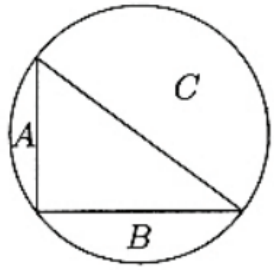
UNIT 4 EXERCISES 21-25

TRIANGLES

1999

21. **(B)** Since $20^2 + 21^2 = 29^2$, the converse of the *Pythagorean Theorem* applies, so the triangle has a right angle. Thus its hypotenuse is a diameter of the circle, so the region with area C is a semicircle and is congruent to the semicircle formed by the other three regions. The area of the triangle is 210, hence $A + B + 210 = C$. To see that the other options are incorrect, note that



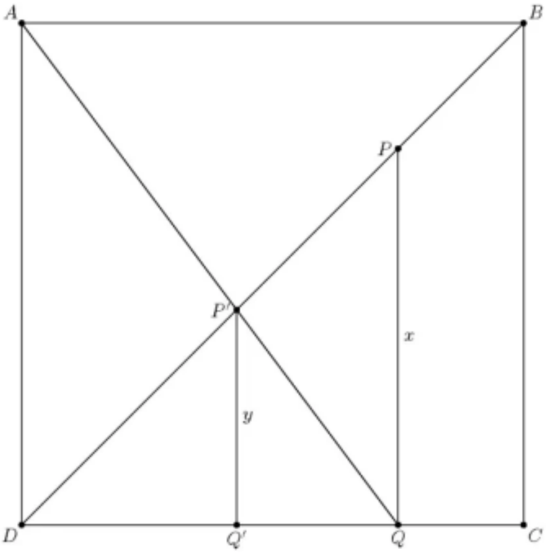
(A) $A + B < A + B + 210 = C$;

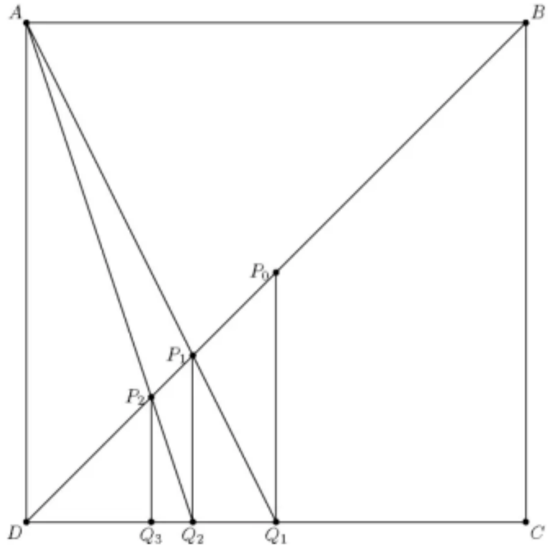
(C) $A^2 + B^2 < (A + B)^2 < (A + B + 210)^2 = C^2$;

(D) $20A + 21B < 29A + 29B < 29(A + B + 210) = 29C$; and

(E) $\dfrac{1}{A^2} + \dfrac{1}{B^2} > \dfrac{1}{A^2} > \dfrac{1}{C^2}$.

2016B 21. **Answer (B):** For any point P between B and D, let Q be the foot of the perpendicular from P to $\overline{CD}$, let P' be the intersection of $\overline{AQ}$ and $\overline{BD}$, and let Q' be the foot of the perpendicular from P' to $\overline{CD}$. Let $x = PQ$ and $y = P'Q'$. Because $\triangle PQD$ and $\triangle P'Q'D$ are isosceles right triangles, $DQ = x$ and $DQ' = y$. Because $\triangle ADQ$ is similar to $\triangle P'Q'Q$, $\frac{1}{x} = \frac{y}{x-y}$. Solving for y gives $y = \frac{x}{1+x}$.





Now let P_0 be the midpoint of $\overline{BD}$. Then $P_0Q_1 = DQ_1 = \frac{1}{2}$. It follows from the analysis above that $P_1Q_2 = DQ_2 = \frac{1}{3}$, $P_2Q_3 = DQ_3 = \frac{1}{4}$, and in general $P_iQ_{i+1} = DQ_{i+1} = \frac{1}{i+2}$. The area of $\triangle DQ_iP_i$ is

$$\frac{1}{2} \cdot DQ_i \cdot P_iQ_{i+1} = \frac{1}{2} \cdot \frac{1}{i+1} \cdot \frac{1}{i+2} = \frac{1}{2}\left(\frac{1}{i+1} - \frac{1}{i+2}\right).$$

The requested infinite sum telescopes:

$$\sum_{i=1}^{\infty} \text{Area of } \triangle DQ_iP_i = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \cdots\right).$$

Its value is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

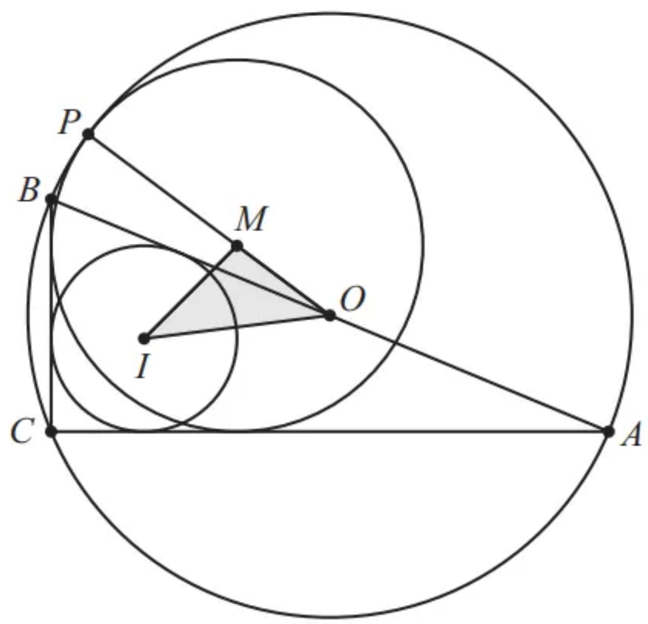
2018B 21. **Answer (E):** Place the figure on coordinate axes with coordinates $A(12, 0)$, $B(0, 5)$, and $C(0, 0)$. The center of the circumscribed circle is the midpoint of the hypotenuse of right triangle ABC, so the coordinates of O are $\left(6, \frac{5}{2}\right)$. The radius r of the inscribed circle equals the area of the triangle divided by its semiperimeter, which here is $30 \div 15 = 2$, so the center of the inscribed circle is $I(2, 2)$. Because the circle with center M is tangent to both coordinate axes, its center has coordinates (ρ, ρ), where ρ is its radius. Let P be the point of tangency of this circle and the circumscribed circle. Then M, P, and O are collinear because $\overline{MP}$ and $\overline{OP}$ are perpendicular to the common tangent line at P. Thus $MO = OP - MP = \frac{13}{2} - \rho$. By the distance formula, $MO = \sqrt{(\rho - 6)^2 + (\rho - \frac{5}{2})^2}$. Equating these expressions and solving for ρ shows that $\rho = 4$. The area of $\triangle MOI$ can now be computed using the shoelace formula:

$$\left| \frac{4 \cdot \frac{5}{2} + 6 \cdot 2 + 2 \cdot 4 - (4 \cdot 6 + \frac{5}{2} \cdot 2 + 2 \cdot 4)}{2} \right| = \frac{7}{2}.$$

Alternatively, the area can be computed as $\frac{1}{2}$ times MI, which by the distance formula is $\sqrt{(4-2)^2 + (4-2)^2} = 2\sqrt{2}$, times the distance from point O to the line MI, whose equation is $x - y + 0 = 0$. This last value is

$$\frac{\left|1 \cdot 6 + (-1) \cdot \frac{5}{2} + 0\right|}{\sqrt{1^2 + (-1)^2}} = \frac{7}{4}\sqrt{2},$$

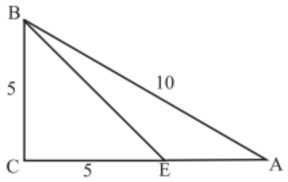
so the area is $\frac{1}{2} \cdot (2\sqrt{2}) \cdot \frac{7}{4}\sqrt{2} = \frac{7}{2}$.



2002A

22. **(C)** Since AB is 10, we have $BC = 5$ and $AC = 5\sqrt{3}$. Choose E on $\overline{AC}$ so that $CE = 5$. Then $BE = 5\sqrt{2}$. For BD to be greater than $5\sqrt{2}$, P has to be inside $\triangle ABE$. The probability that P is inside $\triangle ABE$ is

$$\frac{\text{Area of } \triangle ABE}{\text{Area of } \triangle ABC} = \frac{\frac{1}{2}EA \cdot BC}{\frac{1}{2}CA \cdot BC} = \frac{EA}{AC} = \frac{5\sqrt{3}-5}{5\sqrt{3}} = \frac{\sqrt{3}-1}{\sqrt{3}} = \frac{3-\sqrt{3}}{3}.$$



2007B

22. **Answer (A):** Imagine a third particle that moves in such a way that it is always halfway between the first two. Let D, E, and F denote the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively, and let X, Y, and Z denote the midpoints of $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$, respectively. When the first particle is at A, the second is at D and the third is at X. When the first particle is at F, the second is at C and the third is at Z. Between those two instants, both coordinates of the first two particles are linear functions of time. Because the average of two linear functions is linear, the third particle traverses $\overline{XZ}$. Similarly, the third particle traverses $\overline{ZY}$ as the first traverses $\overline{FB}$ and the second traverses $\overline{CE}$. Finally, as the first particle traverses $\overline{BD}$ and the second traverses $\overline{EA}$, the third traverses $\overline{YX}$. As the first two particles return to A and D, respectively, the third makes a second circuit of $\triangle XYZ$.

If O is the center of $\triangle ABC$, then by symmetry O is also the center of equilateral $\triangle XYZ$. Note that

$$OZ = OC - ZC = \frac{2}{3}CF - \frac{1}{2}CF = \frac{1}{6}CF,$$

so the ratio of the area of $\triangle XYZ$ to that of $\triangle ABC$ is

$$\left(\frac{OZ}{OC}\right)^2 = \left(\frac{\frac{1}{6}CF}{\frac{2}{3}CF}\right)^2 = \frac{1}{16}.$$

2011B

22. **Answer (D):** Let $T_n = \triangle ABC$. Suppose $a = BC$, $b = AC$, and $c = AB$. Because $\overline{BD}$ and $\overline{BE}$ are both tangent to the incircle of $\triangle ABC$, it follows that $BD = BE$. Similarly, $AD = AF$ and $CE = CF$. Then

$$2BE = BE + BD = BE + CE + BD + AD - (AF + CF)$$
$$= a + c - b,$$

that is, $BE = \frac{1}{2}(a + c - b)$. Similarly $AD = \frac{1}{2}(b + c - a)$ and $CF = \frac{1}{2}(a + b - c)$ In the given $\triangle ABC$, suppose that $AB = x+1$, $BC = x-1$, and $AC = x$. Using the formulas for BE, AD, and CF derived before, it must be true that

$$BE = \frac{1}{2}((x-1) + (x+1) - x) = \frac{1}{2}x,$$
$$AD = \frac{1}{2}(x + (x+1) - (x-1)) = \frac{1}{2}x + 1, \text{ and}$$
$$CF = \frac{1}{2}((x-1) + x - (x+1)) = \frac{1}{2}x - 1.$$

Hence both (BC, CA, AB) and (CF, BE, AD) are of the form $(y-1, y, y+1)$. This is independent of the values of a, b, and c, so it holds for all T_n. Furthermore, adding the formulas for BE, AD, and CF shows that the perimeter of T_{n+1} equals $\frac{1}{2}(a+b+c)$, and consequently the perimeter of the last triangle T_N in the sequence is

$$\frac{1}{2^{N-1}}(2011 + 2012 + 2013) = \frac{1509}{2^{N-3}}.$$

The last member T_N of the sequence will fail to define a successor if for the first time the new lengths fail the Triangle Inequality, that is, if

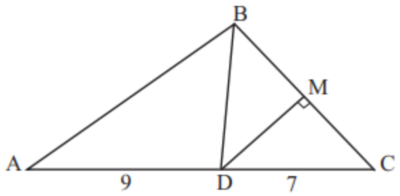
$$-1 + \frac{2012}{2^N} + \frac{2012}{2^N} \le 1 + \frac{2012}{2^N}.$$

Equivalently, $2012 \le 2^{N+1}$ which happens for the first time when $N = 10$. Thus the required perimeter of T_N is $\frac{1509}{2^7} = \frac{1509}{128}$.

2002A

23. **(D)** By the angle-bisector theorem, $\frac{AB}{BC} = \frac{9}{7}$. Let $AB = 9x$ and $BC = 7x$, let $m\angle ABD = m\angle CBD = \theta$, and let M be the midpoint of $\overline{BC}$. Since M is on the perpendicular bisector of $\overline{BC}$, we have $BD = DC = 7$. Then

$$\cos\theta = \frac{\frac{7x}{2}}{7} = \frac{x}{2}.$$



Applying the Law of Cosines to $\triangle ABD$ yields

$$9^2 = (9x)^2 + 7^2 - 2(9x)(7)\left(\frac{x}{2}\right),$$

from which $x = 4/3$ and $AB = 12$. Apply Heron's formula to obtain the area of triangle ABD as $\sqrt{14 \cdot 2 \cdot 5 \cdot 7} = 14\sqrt{5}$.

2002B

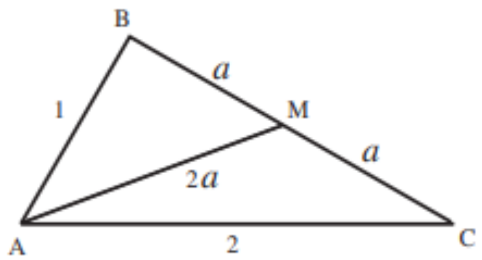
23. **(C)** Let M be the midpoint of $\overline{BC}$, let $AM = 2a$, and let $\theta = \angle AMB$. Then $\cos\angle AMC = -\cos\theta$. Applying the Law of Cosines to $\triangle ABM$ and to $\triangle AMC$ yields, respectively,

$$a^2 + 4a^2 - 4a^2\cos\theta = 1$$

and

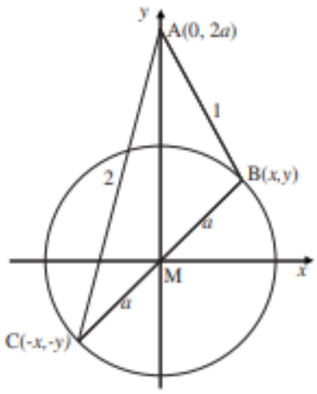
$$a^2 + 4a^2 + 4a^2\cos\theta = 4.$$

Adding, we obtain $10a^2 = 5$, so $a = \sqrt{2}/2$ and $BC = 2a = \sqrt{2}$.



OR

As above, let M be the midpoint of $\overline{BC}$ and $AM = 2a$. Put a rectangular coordinate system in the plane of the triangle with the origin at M so that A has coordinates $(0, 2a)$. If the coordinates of B are (x, y), then the point C has coordinates $(-x, -y)$,



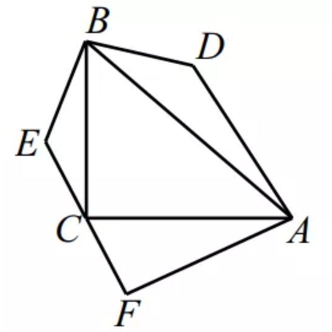
so

$$x^2 + (2a - y)^2 = 1 \quad \text{and} \quad x^2 + (2a + y)^2 = 4.$$

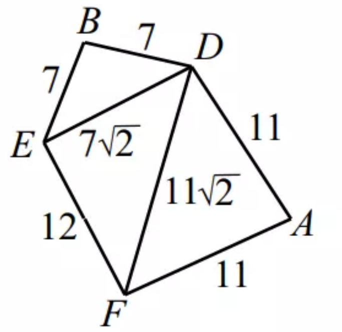
Combining the last two equations gives $2\left(x^2 + y^2\right) + 8a^2 = 5$. But, $x^2 + y^2 = a^2$, so $10a^2 = 5$. Thus, $a = \sqrt{2}/2$ and $BC = \sqrt{2}$.

2006B

23. **(E)** Let D, E, and F be the reflections of P about $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, respectively. Then $\angle FAD = \angle DBE = 90^\circ$, and $\angle ECF = 180^\circ$. Thus the area of pentagon $ADBEF$ is twice that of $\triangle ABC$, so it is s^2.



Observe that $DE = 7\sqrt{2}$, $EF = 12$, and $FD = 11\sqrt{2}$. Furthermore, $(7\sqrt{2})^2 + 12^2 = 98 + 144 = 242 = (11\sqrt{2})^2$, so $\triangle DEF$ is a right triangle. Thus the pentagon can be tiled with three right triangles, two of which are isosceles, as shown.



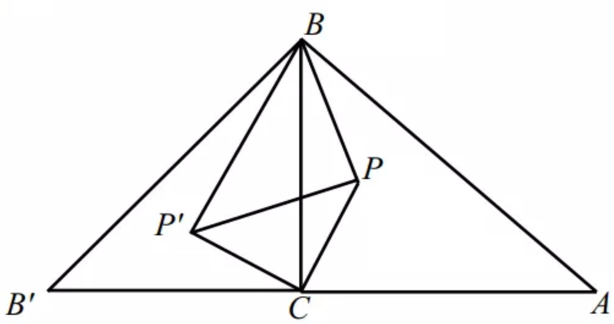
It follows that

$$s^2 = \frac{1}{2} \cdot (7^2 + 11^2) + \frac{1}{2} \cdot 12 \cdot 7\sqrt{2} = 85 + 42\sqrt{2},$$

so $a + b = 127$.

OR

Rotate $\triangle ABC$ 90° counterclockwise about C, and let B' and P' be the images of B and P, respectively.



Then $CP' = CP = 6$, and $\angle PCP' = 90^\circ$, so $\triangle PCP'$ is an isosceles right triangle. Thus $PP' = 6\sqrt{2}$, and $BP' = AP = 11$. Because $\left(6\sqrt{2}\right)^2 + 7^2 = 11^2$, the converse of the Pythagorean Theorem implies that $\angle BPP' = 90^\circ$. Hence $\angle BPC = 135^\circ$. Applying the Law of Cosines in $\triangle BPC$ gives

$$BC^2 = 6^2 + 7^2 - 2 \cdot 6 \cdot 7 \cos 135^\circ = 85 + 42\sqrt{2},$$

and $a + b = 127$.

2007B 23. **Answer (A):** Let the triangle have leg lengths a and b, with $a \leq b$. The given condition implies that

$$\frac{1}{2}ab = 3\left(a + b + \sqrt{a^2 + b^2}\right),$$

so

$$ab - 6a - 6b = 6\sqrt{a^2 + b^2}.$$

Squaring both sides and simplifying yields

$$ab\,(ab - 12a - 12b + 72) = 0,$$

from which

$$(a - 12)(b - 12) = 72.$$

The positive integer solutions of the last equation are $(a, b) = (3, 4)$, $(13, 84)$, $(14, 48)$, $(15, 36)$, $(16, 30)$, $(18, 24)$, and $(20, 21)$. However, the solution $(3, 4)$ is extraneous, and there are six right triangles with the required property.

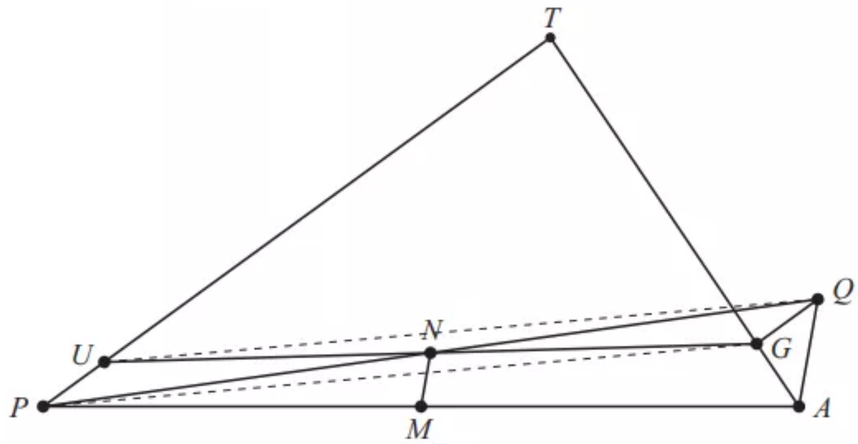
Query: Why do the given conditions imply that the hypotenuse also has integer length?

2018A

23. **Answer (E):** Extend $\overline{PN}$ through N to Q so that $PN = NQ$. Segments $\overline{UG}$ and $\overline{PQ}$ bisect each other, implying that $UPGQ$ is a parallelogram. Therefore $\overline{GQ} \parallel \overline{PT}$, so $\angle QGA = 180° - \angle T = \angle TPA + \angle TAP = 36° + 56° = 92°$. Furthermore $GQ = PU = AG$, so $\triangle QGA$ is isosceles, and $\angle QAG = \frac{1}{2}(180° - 92°) = 44°$. Because $\overline{MN}$ is a midline of $\triangle QPA$, it follows that $\overline{MN} \parallel \overline{AQ}$ and

$$\angle NMP = \angle QAP = \angle QAG + \angle GAP = 44° + 56° = 100°,$$

so acute $\angle NMA = 80°$. (Note that the value of the common length $PU = AG$ is immaterial.)



OR

Place the figure in the coordinate plane with $P = (-5, 0)$, $M = (0, 0)$, $A = (5, 0)$, and T in the first quadrant. Then

$$U = (-5 + \cos 36°, \sin 36°) \quad \text{and} \quad G = (5 - \cos 56°, \sin 56°),$$

and the midpoint N of $\overline{UG}$ is

$$\left(\frac{1}{2}(\cos 36° - \cos 56°), \frac{1}{2}(\sin 36° + \sin 56°)\right).$$

OR

Place the figure in the coordinate plane with $P = (-5, 0)$, $M = (0, 0)$, $A = (5, 0)$, and T in the first quadrant. Then

$$U = (-5 + \cos 36°, \sin 36°) \quad \text{and} \quad G = (5 - \cos 56°, \sin 56°),$$

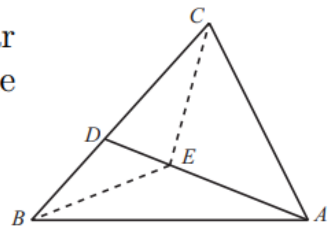
and the midpoint N of $\overline{UG}$ is

$$\left(\frac{1}{2}(\cos 36° - \cos 56°), \frac{1}{2}(\sin 36° + \sin 56°)\right).$$

The tangent of $\angle NMA$ is the slope of line MN, which is calculated as follows using the sum-to-product trigonometric identites:

$$\begin{aligned}\tan(\angle NMA) &= \frac{\sin 36° + \sin 56°}{\cos 36° - \cos 56°} \\ &= \frac{2\sin\frac{36°+56°}{2}\cos\frac{36°-56°}{2}}{-2\sin\frac{36°+56°}{2}\sin\frac{36°-56°}{2}} \\ &= \frac{\cos 10°}{\sin 10°} = \cot 10° = \tan 80°,\end{aligned}$$

2001



24. **(D)** Let E be a point on $\overline{AD}$ such that $\overline{CE}$ is perpendicular to $\overline{AD}$, and draw $\overline{BE}$. Since $\angle ADC$ is an exterior angle of $\triangle ADB$ it follows that

$$\angle ADC = \angle DAB + \angle ABD = 15^\circ + 45^\circ = 60^\circ.$$

Thus, $\triangle CDE$ is a $30^\circ - 60^\circ - 90^\circ$ triangle and $DE = \frac{1}{2}CD = BD$. Hence, $\triangle BDE$ is isosceles and $\angle EBD = \angle BED = 30^\circ$. But $\angle ECB$ is also equal to 30° and therefore $\triangle BEC$ is isosceles with $BE = EC$. On the other hand,

$$\angle ABE = \angle ABD - \angle EBD = 45^\circ - 30^\circ = 15^\circ = \angle EAB.$$

Thus, $\triangle ABE$ is isosceles with $AE = BE$. Hence $AE = BE = EC$. The right triangle AEC is also isosceles with $\angle EAC = \angle ECA = 45^\circ$. Hence,

$$\angle ACB = \angle ECA + \angle ECD = 45^\circ + 30^\circ = 75^\circ.$$

2008A 24. **Answer (D):** Let $C = (0,0)$, $B = (2, 2\sqrt{3})$, and $A = (x, 0)$ with $x > 0$. Then $D = (1, \sqrt{3})$. Let P be on the positive x-axis to the right of A. Then $\angle BAD = \angle PAD - \angle PAB$. Provided $\angle PAD$ and $\angle PAB$ are not right angles, it follows that

$$\begin{aligned}\tan(\angle BAD) = \tan(\angle PAD - \angle PAB) &= \frac{\tan(\angle PAD) - \tan(\angle PAB)}{1 + \tan(\angle PAD)\tan(\angle PAB)} \\ &= \frac{m_{AD} - m_{AB}}{1 + m_{AD}m_{AB}} = \frac{\frac{\sqrt{3}}{1-x} - \frac{2\sqrt{3}}{2-x}}{1 + \frac{\sqrt{3}}{1-x}\cdot\frac{2\sqrt{3}}{2-x}} = \frac{\sqrt{3}x}{x^2 - 3x + 8} \\ &= \frac{\sqrt{3}}{\left(\sqrt{x} - \frac{2\sqrt{2}}{\sqrt{x}}\right)^2 + (4\sqrt{2} - 3)} \le \frac{\sqrt{3}}{4\sqrt{2} - 3},\end{aligned}$$

with equality when $x = 2\sqrt{2}$. If $\angle PAD = 90^\circ$, then

$$\tan(\angle BAD) = \cot(\angle PAB) = \frac{1}{2\sqrt{3}} < \frac{\sqrt{3}}{4\sqrt{2} - 3}.$$

If $\angle PAB = 90^\circ$, then

$$\tan(\angle BAD) = -\cot(\angle PAD) = \frac{1}{\sqrt{3}} < \frac{\sqrt{3}}{4\sqrt{2} - 3}.$$

Therefore the largest possible value of $\tan(\angle BAD)$ is $\sqrt{3}/(4\sqrt{2} - 3)$.

OR

Because the circle with diameter $\overline{BD}$ does not intersect the line AC, it follows that $\angle BAD < 90^\circ$. Thus the value of $\tan(\angle BAD)$ is greatest when $\angle BAD$ is greatest. This occurs when A is placed to minimize the size of the circle passing through A, B, and D, so the maximum is attained when that circle is tangent to $\overline{AC}$ at A. For this location of A, the Power of a Point Theorem implies that

$$AC^2 = CB \cdot CD = 4 \cdot 2 = 8, \text{ and } AC = \sqrt{8} = 2\sqrt{2}.$$

Because $\frac{CA}{CB} = \frac{CD}{CA}$, it follows that $\triangle CAD$ is similar to $\triangle CBA$. Thus $AB = \sqrt{2}AD$. The Law of Cosines, applied to $\triangle ADC$, gives

$$AD^2 = CD^2 + CA^2 - 2CD \cdot CA \cdot \cos 60^\circ = 12 - 4\sqrt{2}.$$

Let O be the center of the circle passing through A, B, and D. The Extended Law of Sines, applied to $\triangle ABD$ and $\triangle ADC$, gives

$$\begin{aligned} 2OB &= \frac{AB}{\sin(\angle BDA)} = \frac{AB}{\sin(\angle ADC)} \\ &= \frac{AB \cdot AD}{AC \cdot \sin 60^\circ} = \frac{2AB \cdot AD}{\sqrt{3}AC} \\ &= \frac{2\sqrt{2}AD^2}{2\sqrt{2}\sqrt{3}} = \frac{AD^2}{\sqrt{3}}.\end{aligned}$$

2008B

24. **Answer (C):** For $n \geq 0$, let $A_n = (a_n, 0)$, and let $c_{n+1} = a_{n+1} - a_n$. Let $B_0 = A_0$, and let $c_0 = 0$. Then for $n \geq 0$,

$$B_{n+1} = \left(a_n + \frac{c_{n+1}}{2}, \frac{\sqrt{3}c_{n+1}}{2} \right),$$

so

$$\left(\frac{\sqrt{3}c_{n+1}}{2} \right)^2 = a_n + \frac{c_{n+1}}{2},$$

from which $3c_{n+1}^2 - 2c_{n+1} - 4a_n = 0$. For $n \geq 1$,

$$B_n = \left(a_n - \frac{c_n}{2}, \frac{\sqrt{3}c_n}{2} \right),$$

so

$$\left(\frac{\sqrt{3}c_n}{2} \right)^2 = a_n - \frac{c_n}{2},$$

from which $3c_n^2 + 2c_n - 4a_n = 0$. Hence $3c_{n+1}^2 - 2c_{n+1} = 4a_n = 3c_n^2 + 2c_n$, and

$$2(c_{n+1} + c_n) = 3(c_{n+1}^2 - c_n^2) = 3(c_{n+1} + c_n)(c_{n-1} - c_n).$$

Thus $c_{n+1} = c_n + \frac{2}{3}$ for $n \geq 0$. It follows that

$$a_n = \frac{2}{3} + \frac{4}{3} + \frac{6}{3} + \cdots + \frac{2n}{3} = \frac{2}{3} \cdot \frac{n(n+1)}{2} = \frac{n(n+1)}{3}.$$

Solving $n(n+1)/3 \geq 100$ gives $n \geq 17$.

2014A 24. **Answer (C):** For integers $n \geq 1$ and $k \geq 0$, if $f_{n-1}(x) = \pm k$, then $f_n(x) = k - 1$. Thus if $f_0(x) = \pm k$, then $f_k(x) = 0$. Furthermore, if $f_n(x) = 0$, then $f_{n+1}(x) = -1$ and $f_{n+2}(x) = 0$. It follows that the zeros of f_{100} are the solutions of $f_0(x) = 2k$ for $-50 \leq k \leq 50$. To count these solutions, note that

$$f_0(x) = \begin{cases} x + 200 & \text{if } x < -100, \\ -x & \text{if } -100 \leq x < 100, \text{ and} \\ x - 200 & \text{if } x \geq 100. \end{cases}$$

The graph of $f_0(x)$ is piecewise linear with turning points at $(-100, 100)$ and $(100, -100)$. The line $y = 2k$ crosses the graph three times for $-49 \leq k \leq 49$ and twice for $k = \pm 50$. Therefore the number of zeros of $f_{100}(x)$ is $99 \cdot 3 + 2 \cdot 2 = 301$.

2011A

25. **Answer (D):** Let

$$100 = qk + r, \text{ with } q, r \in \mathbb{Z} \text{ and } |r| \le \frac{k-1}{2}, \text{ and}$$
$$n = q_1 k + r_1, \text{ with } q_1, r_1 \in \mathbb{Z} \text{ and } |r_1| \le \frac{k-1}{2},$$

so that $\left[\frac{100}{k}\right] = q$ and $\left[\frac{n}{k}\right] = q_1$. Note that $\left[\frac{n+mk}{k}\right] = \left[\frac{n}{k}\right] + m$ for every integer m. Thus n satisfies the required identity if and only if $n + mk$ satisfies the identity for all integers m. Thus all members of a residue class mod k either satisfy the required equality or not; moreover, k divides 99! for every $1 \le k \le 99$, so every residue class mod k in the interval $1 \le n \le 99!$ has the same number of elements. Suppose $r \ge 0$. If $r_1 \ge r - \frac{k-1}{2}$, then

$$100 - n = (q - q_1)\,k + (r - r_1)\,,$$

where $0 \le r - r_1 \le \frac{k-1}{2}$. Thus $\left[\frac{100-n}{k}\right] = q - q_1 = \left[\frac{100}{k}\right] - \left[\frac{n}{k}\right]$. Similarly, if $r_1 < r - \frac{k-1}{2}$, then

$$100 - n = (q - q_1 + 1)\,k + (r - r_1 - k)\,,$$

where $-\frac{k-1}{2} \le r - r_1 - k \le -1$. Thus $\left[\frac{100-n}{k}\right] = q - q_1 + 1 > \left[\frac{100}{k}\right] - \left[\frac{n}{k}\right]$. It follows that the only residue classes r_1 that satisfy the identity are those in the interval $r - \frac{k-1}{2} \le r_1 \le \frac{k-1}{2}$. Thus for $r \ge 0$,

$$P(k) = \frac{1}{k}\left(\frac{k-1}{2} + 1 - \left(r - \frac{k-1}{2}\right)\right) = \frac{k-r}{k} = 1 - \frac{|r|}{k}.$$

Similarly, if $r < 0$ then the identity is satisfied only by the residue classes r_1 in the interval $-\frac{k-1}{2} \le r_1 \le r + \frac{k-1}{2}$. Thus for $r < 0$,

$$P(k) = \frac{1}{k}\left(r + \frac{k-1}{2} + 1 - \left(-\frac{k-1}{2}\right)\right) = \frac{k+r}{k} = 1 - \frac{|r|}{k}.$$

To minimize $P(k)$ in the range $1 \le k \le 99$, where k is odd, first suppose that $r = \frac{k-1}{2}$. Note that $P(k) = \frac{1}{2} + \frac{1}{2k}$, $100 = qk + \frac{k-1}{2}$, and so $201 = k\,(2q+1)$.

The minimum of $P(k)$ in this case is achieved by the largest possible k under this restriction. Because $201 = 3 \cdot 67$, it follows that the largest factor k of 201 in the given range is $k = 67$. In this case $P(67) = \frac{1}{2} + \frac{1}{2 \cdot 67} = \frac{34}{67}$. Second, suppose $r = \frac{1-k}{2}$. In this case $P(k) = \frac{1}{2} + \frac{1}{2k}$ and $199 = k(2q-1)$. Because 199 is prime, it follows that $k = 1$ and $P(k) = 1 > \frac{34}{67}$. Finally, if $|r| \le \frac{k-3}{2}$, then

$$\begin{aligned} P(k) = 1 - \frac{|r|}{k} &> 1 - \frac{k-3}{2k} = \frac{1}{2} + \frac{3}{2k} \\ &\ge \frac{1}{2} + \frac{3}{2 \cdot 99} > \frac{1}{2} + \frac{1}{2 \cdot 67} = \frac{34}{67}. \end{aligned}$$

Therefore the minimum value of $P(k)$ in the required range is $\frac{34}{6}$