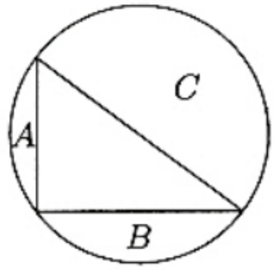
UNIT 3 EXERCISES 21-25

2D GEO WORD PROBLEMS

1999



21. **(B)** Since $20^2 + 21^2 = 29^2$, the converse of the *Pythagorean Theorem* applies, so the triangle has a right angle. Thus its hypotenuse is a diameter of the circle, so the region with area C is a semicircle and is congruent to the semicircle formed by the other three regions. The area of the triangle is 210, hence $A + B + 210 = C$. To see that the other options are incorrect, note that

(A) $A + B < A + B + 210 = C$;

(C) $A^2 + B^2 < (A + B)^2 < (A + B + 210)^2 = C^2$;

(D) $20A + 21B < 29A + 29B < 29(A + B + 210) = 29C$; and

(E) $\dfrac{1}{A^2} + \dfrac{1}{B^2} > \dfrac{1}{A^2} > \dfrac{1}{C^2}$.

2006B

21. **(C)** Let $2a$ and $2b$, respectively, be the lengths of the major and minor axes of the ellipse, and let the dimensions of the rectangle be x and y. Then $x+y$ is the sum of the distances from the foci to point A on the ellipse, which is $2a$. The length of a diagonal of the rectangle is the distance between the foci of the ellipse, which is $2\sqrt{a^2-b^2}$. Thus $x+y=2a$ and $x^2+y^2=4a^2-4b^2$. The area of the rectangle is

$$2006 = xy = \frac{1}{2}\left[(x+y)^2-(x^2+y^2)\right] = \frac{1}{2}\left[(2a)^2-(4a^2-4b^2)\right] = 2b^2,$$

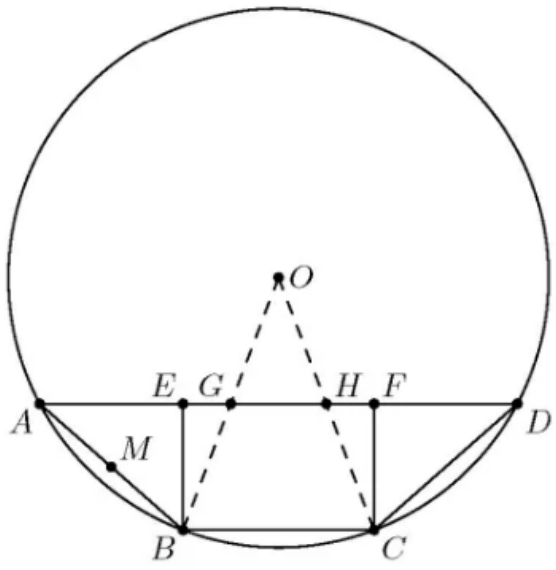
so $b=\sqrt{1003}$. Thus the area of the ellipse is

$$2006\pi = \pi ab = \pi a\sqrt{1003},$$

so $a=2\sqrt{1003}$, and the perimeter of the rectangle is $2(x+y)=4a=8\sqrt{1003}$.

2016A

21. **Answer (E):** Let $ABCD$ be the given cyclic quadrilateral with $AB = BC = CD = 200$, and let E and F be the feet of the perpendicular segments from B and C, respectively, to $\overline{AD}$, as shown in the figure. Let the center of the circle be O, and let $\angle AOB = \angle BOC = \angle COD = \theta$. Because inscribed $\angle BAD$ is half the size of central $\angle BOD = 2\theta$, it follows that $\angle BAD = \theta$. Let M be the midpoint of $\overline{AB}$. Then $\sin\left(\frac{\theta}{2}\right) = \frac{AM}{AO} = \frac{100}{200\sqrt{2}} = \frac{1}{2\sqrt{2}}$. Then $\cos\theta = 1 - 2\sin^2\left(\frac{\theta}{2}\right) = \frac{3}{4}$. Hence $AE = AB\cos\theta = 200 \cdot \frac{3}{4} = 150$, and $FD = 150$ as well. Because $EF = BC = 200$, the remaining side $AD = AE + EF + FD = 150 + 200 + 150 = 500$.



OR

Label the quadrilateral $ABCD$ and the center of the circle as in the first solution. Because the chords $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$ are shorter than the radius, each of $\angle AOB$, $\angle BOC$, and $\angle COD$ is less than 60°, so O is outside the quadrilateral $ABCD$. Let G and H be the intersections of $\overline{AD}$ with $\overline{OB}$ and $\overline{OC}$, respectively. Because $\overline{AD}$ and $\overline{BC}$ are parallel, and $\triangle OAB$ and $\triangle OBC$ are congruent and isosceles, it follows that $\angle ABO = \angle OBC = \angle OGH = \angle AGB$. Thus $\triangle ABG$, $\triangle OGH$, and $\triangle OBC$ are similar and isosceles with $\frac{AB}{BG} = \frac{OG}{GH} = \frac{OB}{BC} = \frac{200\sqrt{2}}{200} = \sqrt{2}$. Then $AG = AB = 200$, $BG = \frac{AB}{\sqrt{2}} = \frac{200}{\sqrt{2}} = 100\sqrt{2}$, and $GH = \frac{OG}{\sqrt{2}} = \frac{BO-BG}{\sqrt{2}} = \frac{200\sqrt{2}-100\sqrt{2}}{\sqrt{2}} = 100$. Therefore $AD = AG + GH + HD = 200 + 100 + 200 = 500$.

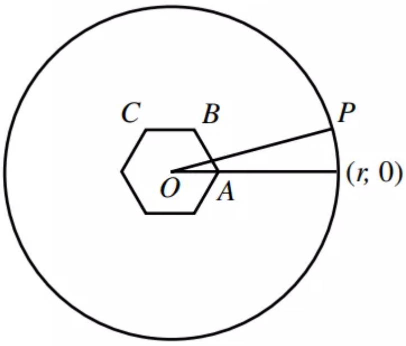
OR

Let θ be the central angle that subtends the side of length 200. Then by the Law of Cosines, $\left(200\sqrt{2}\right)^2 + \left(200\sqrt{2}\right)^2 - 2\left(200\sqrt{2}\right)^2\cos\theta = 200^2$, which gives $\cos\theta = \frac{3}{4}$. The Law of Cosines also gives the square of the fourth side of the quadrilateral as

$$\left(200\sqrt{2}\right)^2 + \left(200\sqrt{2}\right)^2 - 2\left(200\sqrt{2}\right)^2\cos(3\theta)$$

Thus the fourth side has length $\sqrt{250{,}000} = 500$.

2006A 22. **(D)** Place the hexagon in a coordinate plane with center at the origin O and vertex A at $(2,0)$. Let B, C, D, E, and F be the other vertices in counterclockwise order.



Corresponding to each vertex of the hexagon, there is an arc on the circle from which only the two sides meeting at that vertex are visible. The given probability condition implies that those arcs have a combined degree measure of $180°$, so by symmetry each is $30°$. One such arc is centered at $(r,0)$. Let P be the endpoint of this arc in the upper half-plane. Then $\angle POA = 15°$. Side $\overline{BC}$ is visible from points immediately above P, so P is collinear with B and C. Because the perpendicular distance from O to $\overline{BC}$ is $\sqrt{3}$, we have

$$\sqrt{3} = r\sin 15° = r\sin(45° - 30°) = r\left(\sin 45°\cos 30° - \sin 30°\cos 45°\right).$$

So

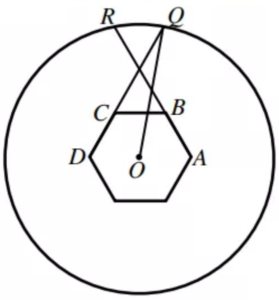
$$\sqrt{3} = r\cdot\frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) = r\cdot\frac{\sqrt{6}-\sqrt{2}}{4}.$$

Therefore

$$r = \frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}} = \frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}}\cdot\frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}+\sqrt{2}} = \sqrt{18}+\sqrt{6} = 3\sqrt{2}+\sqrt{6}.$$

OR

Call the hexagon $ABCDEF$. Side $\overline{AB}$ is visible from point X if and only if X lies in the half-plane that is in the exterior of the hexagon and that is determined by the line AB. The region from which the three sides $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$ are visible is the intersection of three such half-planes.



Let rays AB and DC intersect the circle at R and Q, respectively. Then QR is one of the six arcs of the circle from which three sides are visible. Symmetry implies that the six arcs are congruent, and because the given probability is $1/2$, the measure of each arc is $30°$. Let O be the center of the hexagon and the circle. Then $\angle QOR = 30°$, so

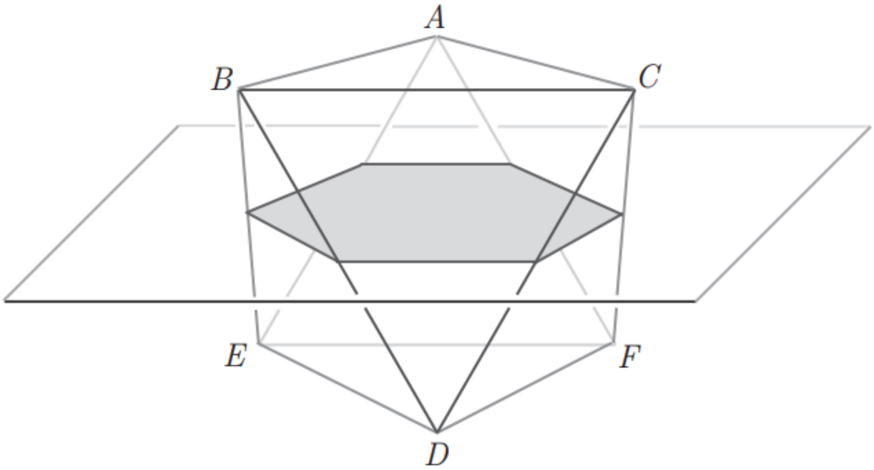
$$\angle QOC = \angle QOR + \frac{1}{2}(\angle BOC - \angle QOR) = 30° + \frac{1}{2}(60° - 30°) = 45°.$$

Thus

$$\angle OQC = 180° - \angle QOC - \angle OCQ = 15°.$$

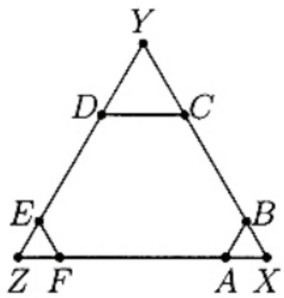
Apply the Law of Sines in $\triangle OQC$ to obtain

2009A 22. **Answer (E):** Let $\triangle ABC$ and $\triangle DEF$ be the two faces of the octahedron parallel to the cutting plane. The plane passes through the midpoints of the six edges of the octahedron that are not sides of either of those triangles. Hence the intersection of the plane with the octahedron is an equilateral hexagon with side length $\frac{1}{2}$. Then by symmetry the hexagon is also equiangular and hence regular. The area of the hexagon is 6 times that of an equilateral triangle with side length $\frac{1}{2}$, so the area is $6\left(\frac{1}{2}\right)^2\frac{\sqrt{3}}{4}=\frac{3\sqrt{3}}{8}$. Therefore $a+b+c=3+3+8=14$.



1999

23. **(E)** Extend $\overline{FA}$ and $\overline{CB}$ to meet at X, $\overline{BC}$ and $\overline{ED}$ to meet at Y, and $\overline{DE}$ and $\overline{AF}$ to meet at Z. The interior angles of the hexagon are 120°. Thus the triangles XYZ, ABX, CDY, and EFZ are equilateral. Since $AB=1$, $BX=1$. Since $CD=2$, $CY=2$. Thus $XY=7$ and $YZ=7$. Since $YD=2$ and $DE=4, EZ=1$. The area of the hexagon can be found by subtracting the areas of the three small triangles from the area of the large triangle:

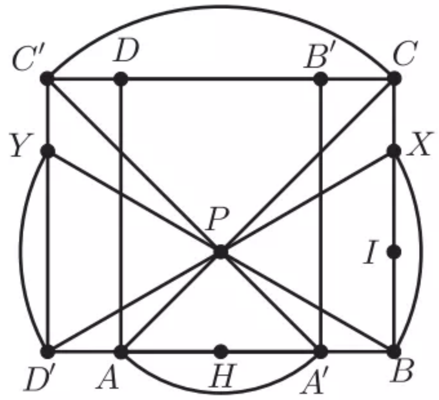


$$7^2\left(\frac{\sqrt{3}}{4}\right)-1^2\left(\frac{\sqrt{3}}{4}\right)-2^2\left(\frac{\sqrt{3}}{4}\right)-1^2\left(\frac{\sqrt{3}}{4}\right)=\frac{43\sqrt{3}}{4}.$$

2007A 23. **Answer (A):** Let $A = (p, \log_a p)$ and $B = (q, 2\log_a q)$. Then $AB = 6 = |p - q|$. Because $\overline{AB}$ is horizontal, $\log_a p = 2\log_a q = \log_a q^2$, so $p = q^2$. Thus $|q^2 - q| = 6$, and the only positive solution is $q = 3$. Note that $C = (q, 3\log_a q)$, so $BC = 6 = \log_a q$, from which $a^6 = q = 3$ and $a = \sqrt[6]{3}$.

2013A **23. Answer (C):**

Assume that the vertices of $ABCD$ are labeled in counterclockwise order. Let A', B', C', and D' be the images of A, B, C, and D, respectively, under the rotation. Because $\triangle A'PA$ and $\triangle C'PC$ are isosceles right triangles, points A' and C' are on lines AB and CD, respectively. Moreover, because $AP = \sqrt{2}$ and $PC = AC - AP = \sqrt{2}(\sqrt{3}+1) - \sqrt{2} = \sqrt{6}$, it follows that $AA' = \sqrt{2}AP = 2$ and $CC' = \sqrt{2}CP = 2\sqrt{3}$. By symmetry, points B' and D' are on lines CD and AB, respectively. Let $X \neq B$ and $Y \neq D'$ be the intersections of $\overline{BC}$ and $\overline{C'D'}$, respectively, with the circle centered at P with radius PB. Note that $PD' = PD = PB$, so this circle also contains D'. Therefore the required region consists of sectors APA', BPX, CPC', and YPD', and triangles BPA', CPX, YPC', and APD'.



Sector APA' has area $\frac{1}{4}\cdot(\sqrt{2})^2\pi = \frac{\pi}{2}$, and sector CPC' has area $\frac{1}{4}\cdot(\sqrt{6})^2\pi = \frac{3\pi}{2}$. Let H and I be the midpoints of $\overline{AA'}$ and $\overline{BX}$, respectively. Then $PH = AH = \frac{\sqrt{2}}{2}AP = 1$, and $PI = HB = AB - AH = \sqrt{3}$. Thus $\triangle BPH$ is a 30-60-90° triangle, implying that $PB = 2$ and $\triangle XPB$ is equilateral. Therefore congruent sectors BPX and YPD' each have area $\frac{1}{6}\cdot 2^2\pi = \frac{2\pi}{3}$.

Congruent triangles BPA' and $D'PA$ each have altitude $PH = 1$ and base $A'B = AB - AH - HA' = \sqrt{3} - 1$, so each has area $\frac{1}{2}(\sqrt{3}-1)$. Congruent triangles CPX and $C'PY$ each have altitude $PI = \sqrt{3}$ and base $XC = BC - BX = \sqrt{3} - 1$, so each has area $\frac{1}{2}(3-\sqrt{3})$.

The area of the entire region is

$$\frac{\pi}{2} + \frac{3\pi}{2} + 2\cdot\frac{2\pi}{3} + 2\left(\frac{\sqrt{3}-1}{2}\right) + 2\left(\frac{3-\sqrt{3}}{2}\right) = \frac{10\pi+6}{3},$$

and $a+b+c = 10+6+3 = 19$.

1999

24. **(B)** Any four of the six given points determine a unique convex quadrilateral, so there are exactly $\binom{6}{4} = 15$ favorable outcomes when the chords are selected randomly. Since there are $\binom{6}{2} = 15$ chords, there are $\binom{15}{4} = 1365$ ways to pick the four chords. So the desired probability is $15/1365 = 1/91$.

1999

24. **(B)** Any four of the six given points determine a unique convex quadrilateral, so there are exactly $\binom{6}{4} = 15$ favorable outcomes when the chords are selected randomly. Since there are $\binom{6}{2} = 15$ chords, there are $\binom{15}{4} = 1365$ ways to pick the four chords. So the desired probability is $15/1365 = 1/91$.

2002B

24. **(E)** We have

$$\text{Area } (ABCD) \le \frac{1}{2} AC \cdot BD,$$

with equality if and only if $AC \perp BD$. Since

$$\begin{aligned} 2002 = \text{Area}(ABCD) &\le \frac{1}{2} AC \cdot BD \\ &\le \frac{1}{2}(AP + PC) \cdot (BP + PD) = \frac{52 \cdot 77}{2} = 2002, \end{aligned}$$

it follows that the diagonals AC and BD are perpendicular and intersect at P. Thus, $AB = \sqrt{24^2 + 32^2} = 40$, $BC = \sqrt{28^2 + 32^2} = 4\sqrt{113}$, $CD = \sqrt{28^2 + 45^2} = 53$, and $DA = \sqrt{45^2 + 24^2} = 51$. The perimeter of $ABCD$ is therefore

$$144 + 4\sqrt{113} = 4\left(36 + \sqrt{113}\right).$$

2011A

24. **Answer (B):** Factoring or using the quadratic formula with z^4 as the variable yields $P(z) = (z^4 - 1)(z^4 + (4\sqrt{3} + 7))$. Moreover, $4\sqrt{3} + 7 = (\sqrt{3} + 2)^2$ and $2(\sqrt{3}+2) = 2\sqrt{3}+4 = (\sqrt{3}+1)^2$; thus $4\sqrt{3}+7 = (\frac{1}{2}(\sqrt{6}+\sqrt{2}))^4$. If $w = \frac{1}{2}(\sqrt{3}+1)$, then the eight zeros of $P(z)$ are $1, -1, i, -i, w(1+i), w(-1+i), w(-1-i)$, and $w(1-i)$.

The distances from 1 to the other zeros are

$$|1 - (-1)| = 2, |1 \pm i| = \sqrt{2}, |1 - w(1 \pm i)| = \sqrt{(1-w)^2 + w^2} = \sqrt{2}, \text{ and}$$
$$|1 - w(-1 \pm i)| = \sqrt{(1+w)^2 + w^2} = \sqrt{2\sqrt{3} + 4} = \sqrt{3} + 1.$$

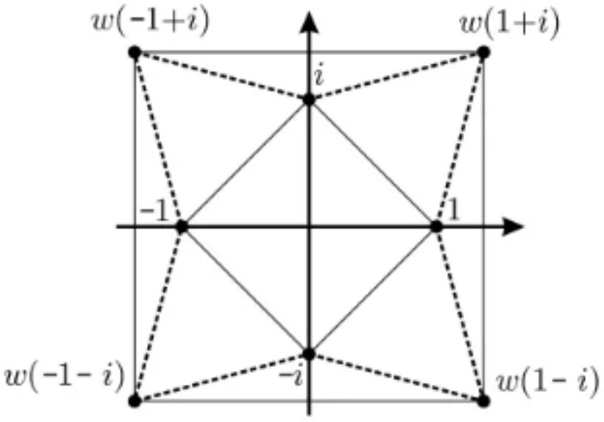
Similarly, the distances from $w(1+i)$ to the other zeros are

$$|w(1+i) - w(1-i)| = |w(1+i) - w(-1+i)| = 2w = \sqrt{3} + 1,$$
$$|w(1+i) - w(-1-i)| = 2\sqrt{2}w = \sqrt{6} + \sqrt{2},$$

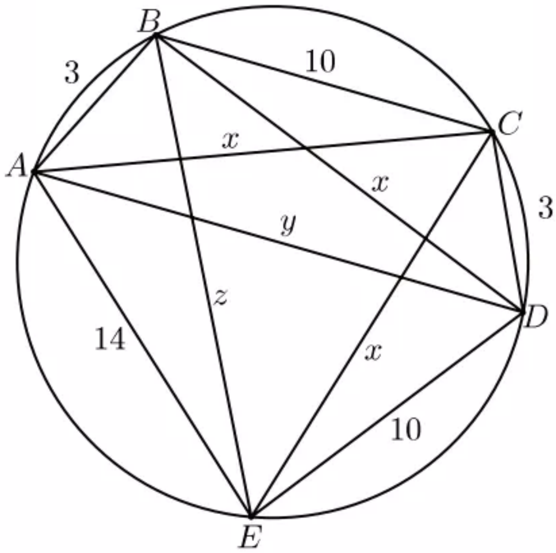
and by symmetry,

$$|w(1+i) - 1| = |w(1+i) - i| = \sqrt{2}, \text{ and}$$
$$|w(1+i) + 1| = |w(1+i) + i| = \sqrt{3} + 1.$$

Because the set of zeros is 4-fold symmetric with respect to the origin, it follows that every line segment joining two of the zeros has length at least $\sqrt{2}$. This shows that any polygon with vertices at the zeros has perimeter at least $8\sqrt{2}$. Finally, note that the polygon with consecutive vertices $1, w(1+i), i, w(-1+i), -1, w(-1-i), -i$, and $w(1-i)$ has perimeter $8\sqrt{2}$.



2014B 24. **Answer (D):** Let $x = AC$, $y = AD$, and $z = BE$. Because the arcs ABC, BCD, and CDE are congruent, it follows that $AC = BD = CE = x$.



By Ptolemy's Theorem applied to the quadrilaterals $ABCD$, $ABDE$, and $BCDE$, it follows that

$$10y + 9 = x^2, \ 30 + 14x = yz, \text{ and } 100 + 3z = x^2.$$

Solving for y and z in the first and third equations and substituting in the second equation gives

$$30 + 14x = \left(\frac{x^2 - 9}{10}\right)\left(\frac{x^2 - 100}{3}\right) = \frac{x^4 - 109x^2 + 900}{30},$$

which implies that

$$900 + 420x = x^4 - 109x^2 + 900.$$

Thus $x^3 - 109x - 420 = 0$. This equation factors as $(x - 12)(x + 5)(x + 7) = 0$. Because $x > 0$ it follows that $x = 12$, $y = \frac{1}{10}(x^2 - 9) = \frac{135}{10} = \frac{27}{2}$, and $z = \frac{1}{3}(x^2 - 100) = \frac{44}{3}$. The required sum of diagonals equals $3x + y + z = \frac{385}{6}$, so $m + n = 385 + 6 = 391$.

2015B

24. **Answer (D):** Points A, B, C, D, and R all lie on the perpendicular bisector of $\overline{PQ}$. Assume R lies between A and B. Let $y = AR$ and $x = \frac{AP}{5}$. Then $BR = 39 - y$ and $BP = 8x$, so $y^2 + 24^2 = 25x^2$ and $(39 - y)^2 + 24^2 = 64x^2$. Subtracting the two equations gives $x^2 = 39 - 2y$, from which $y^2 + 50y - 399 = 0$, and the only positive solution is $y = 7$. Thus $AR = 7$, and $BR = 32$.

Note that circles A and B are determined by the assumption that R lies between A and B. Thus because the four circles are noncongruent, R does not lie between C and D. Let $w = CR$ and $z = \frac{CP}{5}$. Then $DR = 39 + w$ and $DP = 8z$, so $w^2 + 24^2 = 25z^2$ and $(39 + w)^2 + 24^2 = 64z^2$. Subtracting the two equations gives $z^2 = 39 + 2w$, from which $w^2 - 50w - 399 = 0$, and the only positive solution is $w = 57$. Thus $CR = 57$ and $DR = 96$. Again, the uniqueness of the solution implies that R must indeed lie between A and B.

The requested sum is $7 + 32 + 57 + 96 = 192$.

2017A

24. **Answer (A):** Because $\overline{YE}$ and $\overline{EF}$ are parallel to $\overline{AD}$ and $\overline{AC}$, respectively, $\triangle XEY \sim \triangle XAD$ and $\triangle XEF \sim \triangle XAC$. Therefore

$$\frac{XY}{XE} = \frac{XD}{XA} \quad \text{and} \quad \frac{XF}{XE} = \frac{XC}{XA}.$$

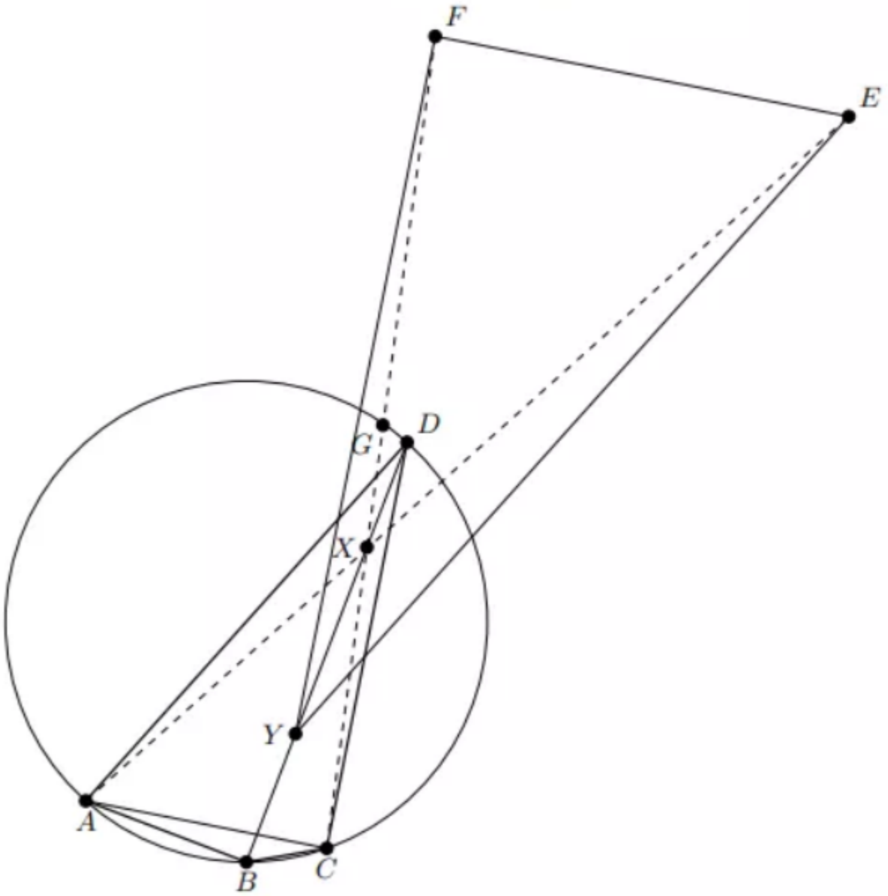
It follows that

$$\frac{XC}{XD} = \frac{XF}{XY}.$$

The Power of a Point Theorem applied to circle O and point X implies that $XC \cdot XG = XD \cdot XB$. Together with the previous equation this

implies that $XF \cdot XG = XB \cdot XY$. Let $d = BD$; then $DX = \frac{1}{4}d$ and $BY = \frac{11}{36}d$. It follows that

$$\begin{aligned} XF \cdot XG = XB \cdot XY &= (BD - DX) \cdot (BD - DX - BY) \\ &= \left(d - \frac{1}{4}d\right)\left(d - \frac{1}{4}d - \frac{11}{36}d\right) \\ &= \frac{3}{4} \cdot \frac{4}{9}d^2 = \frac{d^2}{3}. \end{aligned}$$



To determine d, note that because $ABCD$ is a cyclic quadrilateral it follows that $\alpha = \angle BAD = \pi - \angle DCB$. Applying the Law of Cosines to $\triangle ABD$ and $\triangle BCD$ yields

Therefore

$$\frac{73 - d^2}{48} = \frac{d^2 - 40}{24},$$

2017B

24. **Answer (D):** Let F lie on $\overline{AB}$ so that $\overline{DF} \perp \overline{AB}$. Because $BCDF$ is a rectangle, $\angle FCB \cong \angle DBC \cong \angle CAB \cong \angle BCE$, so E lies on $\overline{CF}$ and it is the foot of the altitude to the hypotenuse in $\triangle CBF$. Therefore $\triangle BEF \sim \triangle CBF \cong \triangle BCD \sim \triangle ABC$. Because

$$\overline{DF} \perp \overline{AB}, \quad \overline{FE} \perp \overline{EB}, \quad \text{and} \quad \frac{AB}{DF} = \frac{AB}{BC} = \frac{BE}{FE},$$

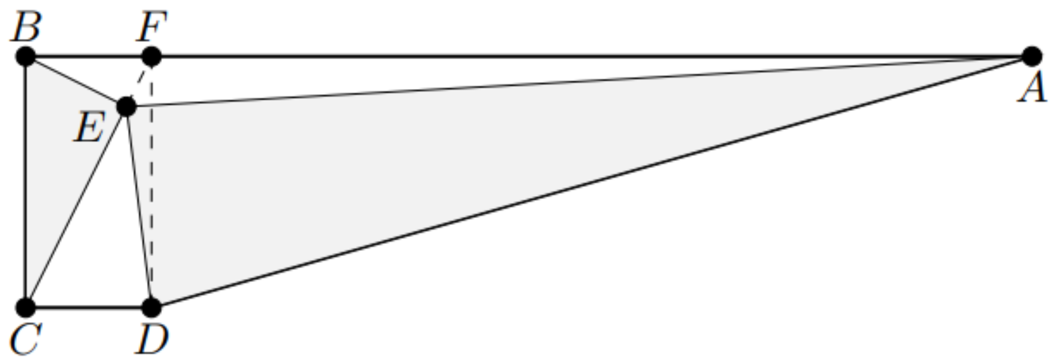
it follows that $\triangle ABE \sim \triangle DFE$. Thus $\angle DEA = \angle DEF - \angle AEF = \angle AEB - \angle AEF = \angle FEB = 90^\circ$. Furthermore,

$$\frac{AE}{ED} = \frac{BE}{EF} = \frac{AB}{BC},$$

so $\triangle AED \sim \triangle ABC$. Assume without loss of generality that $BC = 1$, and let $AB = r > 1$. Because $\frac{AB}{BC} = \frac{BC}{CD}$, it follows that $BF = CD = \frac{1}{r}$. Then

$$17 = \frac{\text{Area}(\triangle AED)}{\text{Area}(\triangle CEB)} = AD^2 = FD^2 + AF^2 = 1 + \left(r - \frac{1}{r}\right)^2,$$

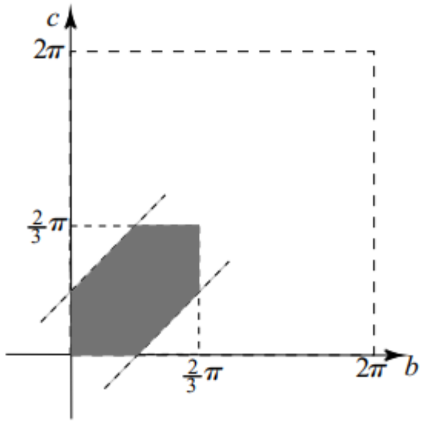
and because $r > 1$ this yields $r^2 - 4r - 1 = 0$, with positive solution $r = 2 + \sqrt{5}$.



2003B 25. (D) We can assume that the circle has its center at $(0,0)$ and a radius of 1. Call the three points A, B, and C, and let a, b, and c denote the length of the counterclockwise arc from $(1,0)$ to A, B, and C, respectively. Rotating the circle if necessary, we can also assume that $a=\pi/3$. Since b and c are chosen at random from $[0,2\pi)$, the ordered pair (b,c) is chosen at random from a square with area $4\pi^2$ in the bc-plane. The condition of the problem is met if and only if

$$0<b<\frac{2\pi}{3},\quad 0<c<\frac{2\pi}{3},\quad \text{and}\quad |b-c|<\frac{\pi}{3}.$$

This last inequality is equivalent to $b-\dfrac{\pi}{3}<c<b+\frac{\pi}{3}$.



The graph of the common solution to these inequalities is the shaded region shown. The area of this region is

$$\left(\frac{6}{8}\right)\left(\frac{2\pi}{3}\right)^2=\pi^2/3,$$

so the requested probability is

$$\frac{\pi^2/3}{4\pi^2}=\frac{1}{12}.$$

2007B 25. **Answer (C):** Introduce a coordinate system in which $D = (-1, 0, 0)$, $E = (1, 0, 0)$, and $\triangle ABC$ lies in a plane $z = k > 0$. Because $\angle CDE$ and $\angle DEA$ are right angles, A and C are located on circles of radius 2 centered at E and D in the planes $x = 1$ and $x = -1$, respectively. Thus $A = (1, y_1, k)$ and $C = (-1, y_2, k)$, where $y_j = \pm\sqrt{4 - k^2}$ for $j = 1$ and 2. Because $AC = 2\sqrt{2}$, it follows that $(1 - (-1))^2 + (y_1 - y_2)^2 = \left(2\sqrt{2}\right)^2$. If $y_1 = y_2$, there is no solution, so $y_1 = -y_2$. It may be assumed without loss of generality that $y_1 > 0$, in which case $y_1 = 1$ and $y_2 = -1$. It follows that $k = \sqrt{3}$, so $A = (1, 1, \sqrt{3})$, $C = \left(-1, -1, \sqrt{3}\right)$, and B is one of the points $\left(1, -1, \sqrt{3}\right)$ or $\left(-1, 1, \sqrt{3}\right)$. In the first case, $BE = 2$ and $\overline{BE} \perp \overline{DE}$. In the second case, $BD = 2$ and $\overline{BD} \perp \overline{DE}$. In either case, the area of $\triangle BDE$ is $(1/2)(2)(2) = 2$.

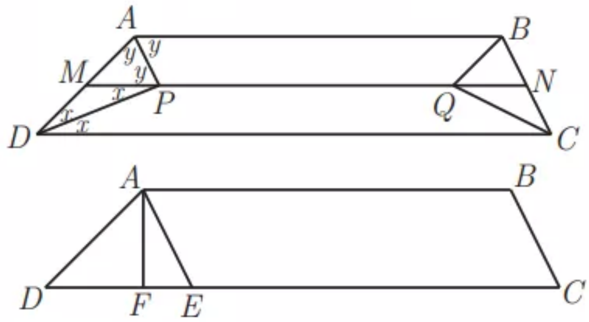
2008B 25. **Answer (B):** Let M and N be the midpoints of sides AD and BC. Set $\angle BAD = 2y$ and $\angle ADC = 2x$. We have $x+y = 90^\circ$, from which it follows that $\angle APD = 90^\circ$. Hence in triangle APD, MP is the median to the hypotenuse AD, so $AM = MD = MP$ and $\angle MPA = \angle MAP = \angle PAB$. Thus, $\overline{MP} \parallel \overline{AB}$. Likewise, $\overline{QN} \parallel \overline{AB}$. It follows that M, P, Q, and N are collinear, and

$$PQ = MN - MP - QN = \frac{AB + CD - AD - BC}{2} = 9.$$

The area of $ABQCDP$ is equal to the sum of the areas of two trapezoids $ABQP$ and $CDPQ$. Let F be the foot of the perpendicular from A to $\overline{CD}$. Then the area of $ABQCDP$ is equal to

$$\frac{AB + PQ}{2} \cdot \frac{AF}{2} + \frac{CD + PQ}{2} \cdot \frac{AF}{2} = 12AF.$$

Let E lie on $\overline{DC}$ so that $\overline{AE} \parallel \overline{BC}$. Then $AE = BC = 5$ and $DE = CD - CE = CD - AB = 8$. We have $AD^2 - DF^2 = AF^2 = AE^2 - EF^2 = AE^2 - (DE - DF)^2$, or $49 - DF^2 = 25 - (8 - DF)^2$. Solving the last equation gives $DF = \frac{11}{2}$. Thus $AF = \frac{5\sqrt{3}}{2}$ and the area of $ABQCDP$ is $12AF = 30\sqrt{3}$.



OR

As in the first solution, conclude that $AE = 5$ and $DE = 8$. Apply the Law of Cosines to $\triangle ADE$ to obtain

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$$\cos(\angle AED) = \frac{8^2 + 5^2 - 7^2}{2 \cdot 8 \cdot 5} = \frac{1}{2}.$$

Therefore $\angle AED = 60^\circ$, so $AF = 5\sqrt{3}/2$, and the area of $ABQCDP$ is $30\sqrt{3}$.

2010A 25. **Answer (C):** Suppose that a quadrilateral with sides $a \geq b \geq c \geq d$ and with perimeter 32 exists. By the triangle inequality $a < b+c+d = 32-a$, so $a \leq 15$. Reciprocally, if (a,b,c,d) is a quadruple of positive integers whose sum equals 32, and whose maximum entry is $a \leq 15$, then $b+c+d = 32-a \geq 17 > a$, so the triangle inequality is satisfied. This is the only condition required to guarantee the existence of a convex quadrilateral with given side lengths. Moreover, if the cyclic order of the sides is specified, then there is exactly one such cyclic quadrilateral.

The problem reduces to counting all the quadruples (a,b,c,d) of positive integers with $a+b+c+d = 32$, $\max(a,b,c,d) \leq 15$, and where two quadruples are considered the same if they generate the same quadrilateral, that is if one is a cyclic permutation of the other one. For example $(12,4,5,11)$ and $(5,11,12,4)$ generate the same quadrilateral.

The number of quadruples (a,b,c,d) with $a+b+c+d = 32$ can be counted as follows: consider 31 spots on a line to be filled with 28 ones and 3 plus signs. There are $\binom{31}{3}$ ways to choose the locations of the plus signs, and every such assignment is in one-to-one correspondence to the quadruple (a',b',c',d'), where each entry indicates the number of ones between consecutive plus signs. Setting $(a,b,c,d) = (1,1,1,1)+(a',b',c',d')$ gives precisely all quadruples where $a,b,c,d \geq 1$ and $a+b+c+d = 32$. To count those where the maximum entry is 16 or more, consider 13 ones and 3 plus signs. There are $\binom{16}{3}$ quadruples (a',b',c',d') where $a',b',c',d' \geq 0$ and $a'+b'+c'+d' = 13$, there are 4 ways to choose one of the coordinates, say a', to be the maximum. Then the quadruple $(a,b,c,d) = (16,1,1,1)+(a',b',c',d')$ satisfies our requirements. Thus there are exactly $4\binom{16}{3}$ quadruples (a,b,c,d) where $a,b,c,d \geq 1$, $a+b+c+d = 32$, and $\max(a,b,c,d) \geq 16$; consequently, there are

$$\binom{31}{3} - 4\binom{16}{3} \qquad (1)$$

quadruples (a,b,c,d) where $a,b,c,d \geq 1$, $a+b+c+d = 32$, and $\max(a,b,c,d) \leq 15$.

If (a,b,c,d) consists of distinct entries, then it has exactly 4 cyclic permutations. The same occurs if only two entries are equal to each other, or three entries are equal to each other and the remaining entry is not. If (a,b,c,d) has two pairs of entries equal to each other ordered (a,a,b,b), then it has 4 cyclic permutations, but if they are ordered (a,b,a,b) then it has only 2 cyclic permutations. Finally, if all entries are equal then there is only one cyclic permutation.

There are exactly $2 \cdot 7 = 14$ quadruples of the form (a,b,a,b) with $a \neq b$ and $a+b = 16$ and there is only one quadruple $(a,a,a,a) = (8,8,8,8)$ with four equal entries. Adding to (**??**) the number of quadruples of the form (a,b,a,b) and 3 times the number of quadruples of the form (a,a,a,a), guarantees that all classes of equivalence under cyclic permutations are counted exactly 4 times. Therefore the required number of cyclic quadrilaterals is

$$\frac{1}{4}\left(\binom{31}{3} - 4\binom{16}{3} + 14 + 3\right) = \frac{1}{4}(31 \cdot 5 \cdot 29 - 32 \cdot 5 \cdot 14 + 17)$$
$$= \frac{1}{4}(5 \cdot 451 + 17) = 568.$$

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26. **(D)** The interior angle of a regular n-gon is $180(1-2/n)$. Let a be the number of sides of the congruent polygons and let b be the number of sides of the third polygon (which could be congruent to the first two polygons). Then

$$2 \cdot 180\left(1-\frac{2}{a}\right)+180\left(1-\frac{2}{b}\right)=360.$$

Clearing denominators and factoring yields the equation

$$(a-4)(b-2)=8,$$

whose four positive integral solutions are $(a, b) = (5, 10), (6, 6), (8, 4)$, and $(12, 3)$. These four solutions give rise to polygons with perimeters of $14, 12, 14$ and 21, respectively, so the largest possible perimeter is 21.