UNIT 2 EXERCISES 21-25

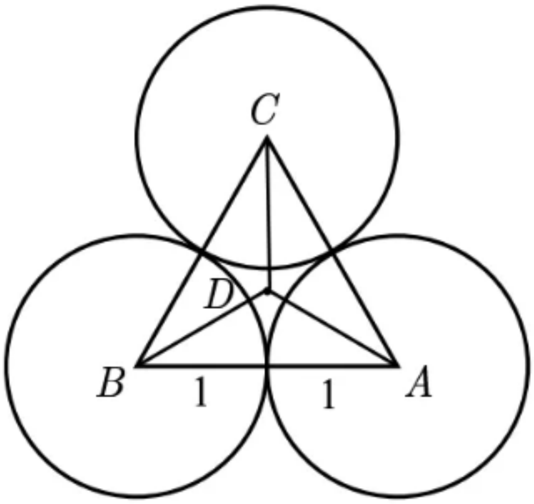
3D GEO

2004A 22. **(B)** Let A, B, C and E be the centers of the three small spheres and the large sphere, respectively. Then $\triangle ABC$ is equilateral with side length 2. If D is the intersection of the medians of $\triangle ABC$, then E is directly above D. Because $AE = 3$ and $AD = 2\sqrt{3}/3$, it follows that

$$DE = \sqrt{3^2 - \left(\frac{2\sqrt{3}}{3}\right)^2} = \frac{\sqrt{69}}{3}.$$

Because D is 1 unit above the plane and the top of the larger sphere is 2 units above E, the distance from the plane to the top of the larger sphere is

$$3 + \frac{\sqrt{69}}{3}.$$



2005A

22. **(B)** Let the dimensions of P be x, y, and z. The sum of the lengths of the edges of P is $4(x+y+z)$, and the surface area of P is $2xy+2yz+2xz$, so

$$x+y+z=28 \quad \text{and} \quad 2xy+2yz+2xz=384.$$

Each internal diagonal of P is a diameter of the sphere, so

$$(2r)^2=(x^2+y^2+z^2)=(x+y+z)^2-(2xy+2xz+2yz)=28^2-384=400.$$

So $2r=20$ and $r=10$.

Note: There are infinitely many positive solutions of the system $x+y+z=28$, $2xy+2yz+2xz=384$, so there are infinitely many non-congruent boxes meeting the given conditions, but each can be inscribed in a sphere of radius 10.

2015B

23. **Answer (B):** Because the volume and surface area are numerically equal, $abc=2(ab+ac+bc)$. Rewriting the equation as $ab(c-6)+ac(b-6)+bc(a-6)=0$ shows that $a \le 6$. The original equation can also be written as $(a-2)bc-2ab-2ac=0$. Note that if $a=2$, this becomes $b+c=0$, and there are no solutions. Otherwise, multiplying both sides by $a-2$ and adding $4a^2$ to both sides gives $[(a-2)b-2a][(a-2)c-2a]=4a^2$. Consider the possible values of a.

$a=1$: $(b+2)(c+2)=4$
There are no solutions in positive integers.

$a=3$: $(b-6)(c-6)=36$
The 5 solutions for (b,c) are $(7,42)$, $(8,24)$, $(9,18)$, $(10,15)$, and $(12,12)$.

$a=4$: $(b-4)(c-4)=16$
The 3 solutions for (b,c) are $(5,20)$, $(6,12)$, and $(8,8)$.

$a=5$: $(3b-10)(3c-10)=100$
Each factor must be congruent to 2 modulo 3, so the possible pairs of factors are $(2,50)$ and $(5,20)$. The solutions for (b,c) are $(4,20)$ and $(5,10)$, but only $(5,10)$ has $a \le b$.

$a=6$: $(b-3)(c-3)=9$
The solutions for (b,c) are $(4,12)$ and $(6,6)$, but only $(6,6)$ has $a \le b$.

Thus in all there are 10 ordered triples (a,b,c): $(3,7,42)$, $(3,8,24)$, $(3,9,18)$, $(3,10,15)$, $(3,12,12)$, $(4,5,20)$, $(4,6,12)$, $(4,8,8)$, $(5,5,10)$, and $(6,6,6)$.

2018B

23. **Answer (C):** To travel from A to B, one could circle $135°$ east along the equator and then $45°$ north. Construct an x-y-z coordinate system with origin at Earth's center C, the positive x-axis running through A, the positive y-axis running through the equator at $160°$ west longitude, and the positive z-axis running through the North Pole. Set Earth's radius to be 1. The coordinates of A are $(1, 0, 0)$. Let b be the y-coordinate of B; note that $b > 0$. Then the x-coordinate of B will be $-b$, and the z-coordinate will be $\sqrt{2}b$. Because the distance from the center of Earth is 1,

$$\sqrt{(-b)^2 + b^2 + \left(\sqrt{2}b\right)^2} = 1,$$

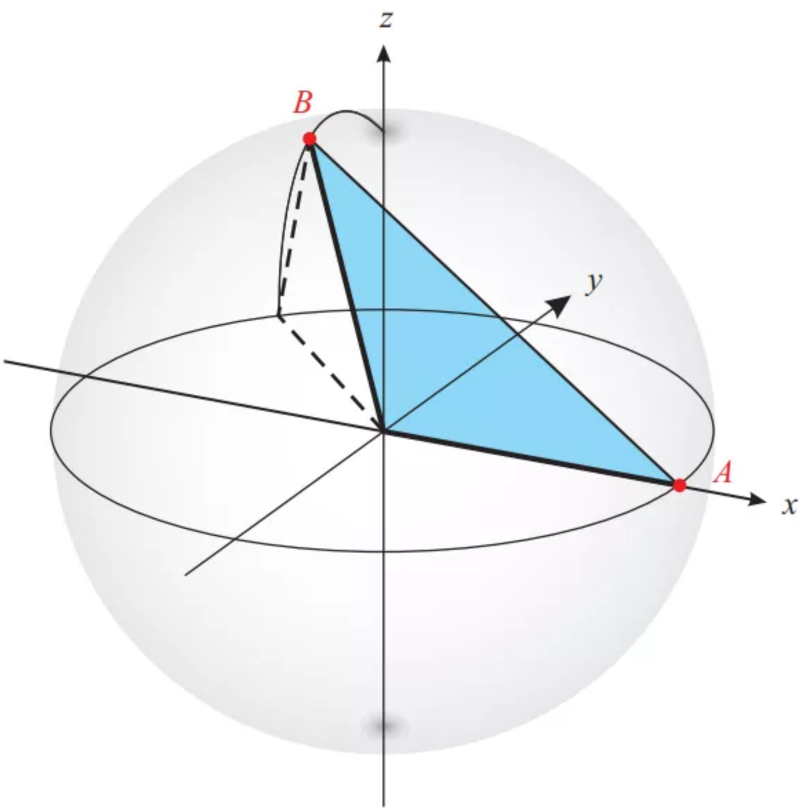
so $b = \frac{1}{2}$, and the coordinates are $\left(-\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right)$. The distance AB is therefore

$$\sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{3}.$$

Applying the Law of Cosines to $\triangle ACB$ gives

$$3 = 1 + 1 - 2 \cdot 1 \cdot 1 \cdot \cos\angle ACB,$$

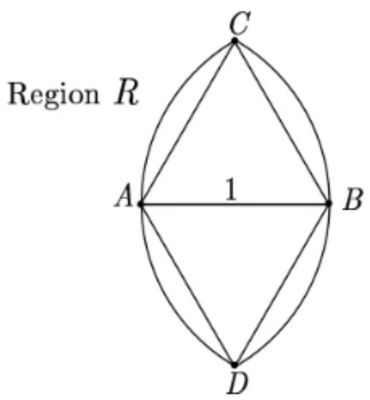
so $\cos\angle ACB = -\frac{1}{2}$ and $\angle ACB = 120°$. An alternative to using the Law of Cosines to find $\cos\angle ACB$ is to compute the dot product of the unit vectors $(1, 0, 0)$ and $\left(-\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right)$.



2004A

24. **(C)** The center of the disk lies in a region R, consisting of all points within 1 unit of both A and B. Let C and D be the points of intersection of the circles of radius 1 centered at A and B. Because $\triangle ABC$ and $\triangle ABD$ are equilateral, arcs CAD and CBD are each 120°. Thus the sector bounded by $\overline{BC}$, $\overline{BD}$, and arc CAD has area $\pi/3$, as does the sector bounded by $\overline{AC}$, $\overline{AD}$, and arc CBD. The intersection of the two sectors, which is the union of the two triangles, has area $\sqrt{3}/2$, so the area of R is

$$\frac{2\pi}{3} - \frac{\sqrt{3}}{2}.$$

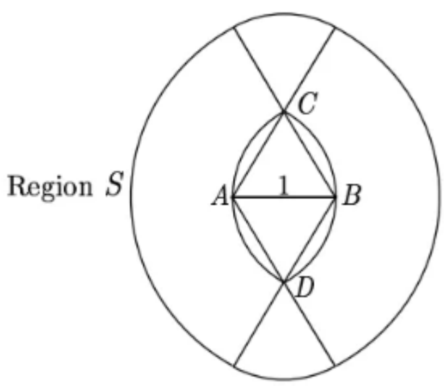


The region S consists of all points within 1 unit of R. In addition to R itself, S contains two 60° sectors of radius 1 and two 120° annuli of outer radius 2 and inner radius 1. The area of each sector is $\pi/6$, and the area of each annulus is

$$\frac{\pi}{3}(2^2 - 1^2) = \pi.$$

Therefore the area of S is

$$\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) + 2\left(\frac{\pi}{6} + \pi\right) = 3\pi - \frac{\sqrt{3}}{2}.$$



1999 25. **(B)** Multiply both sides of the equation by 7! to obtain

$$3600 = 2520a_2 + 840a_3 + 210a_4 + 42a_5 + 7a_6 + a_7.$$

It follows that $3600 - a_7$ is a multiple of 7, which implies that $a_7 = 2$. Thus,

$$\frac{3598}{7} = 514 = 360a_2 + 120a_3 + 30a_4 + 6a_5 + a_6.$$

Reason as above to show that $514 - a_6$ is a multiple of 6, which implies that $a_6 = 4$. Thus, $510/6 = 85 = 60a_2 + 20a_3 + 5a_4 + a_5$. Then it follows that $85 - a_5$ is a multiple of 5, whence $a_5 = 0$. Continue in this fashion to obtain $a_4 = 1, a_3 = 1$, and $a_2 = 1$. Thus the desired sum is $1 + 1 + 1 + 0 + 4 + 2 = 9$.

2015B 25. **Answer (B):** Modeling the bee's path with complex numbers, set $P_0 = 0$ and $z = e^{\pi i/6}$. It follows that for $j \geq 1$,

$$P_j = \sum_{k=1}^{j} k z^{k-1}.$$

Thus

$$P_{2015} = \sum_{k=0}^{2015} k z^{k-1} = \sum_{k=0}^{2014} (k+1)\, z^k = \sum_{k=0}^{2014} \sum_{j=0}^{k} z^k.$$

Interchanging the order of summation and summing the geometric series gives

$$\begin{aligned} P_{2015} &= \sum_{j=0}^{2014} \sum_{k=j}^{2014} z^k = \sum_{j=0}^{2014} z^j \sum_{k=0}^{2014-j} z^k \\ &= \sum_{j=0}^{2014} \frac{z^j\left(z^{2015-j} - 1\right)}{z-1} = \sum_{j=0}^{2014} \frac{z^{2015} - z^j}{z-1} = \frac{1}{z-1} \sum_{j=0}^{2014} \left(z^{2015} - z^j\right) \\ &= \frac{1}{z-1}\left(2015 z^{2015} - \sum_{j=0}^{2014} z^j\right) = \frac{1}{z-1}\left(2015 z^{2015} - \frac{z^{2015}-1}{z-1}\right) \\ &= \frac{1}{(z-1)^2}\left(2015 z^{2015}(z-1) - z^{2015} + 1\right) \\ &= \frac{1}{(z-1)^2}\left(2015 z^{2016} - 2016 z^{2015} + 1\right). \end{aligned}$$

Note that $z^{12} = 1$ and thus $z^{2016} = (z^{12})^{168} = 1$ and $z^{2015} = \frac{1}{z}$. It follows that

$$P_{2015} = \frac{2016}{(z-1)^2}\left(1 - \frac{1}{z}\right) = \frac{2016}{z(z-1)}.$$

Finally,

$$|z-1|^2 = \left|\cos\left(\frac{\pi}{6}\right) - 1 + i \sin\left(\frac{\pi}{6}\right)\right|^2 = \left|\frac{\sqrt{3}}{2} - 1 + \frac{i}{2}\right|^2 = 2 - \sqrt{3} = \frac{(\sqrt{3}-1)^2}{2},$$

1999 29. **(C)** Let A, B, C, and D be the vertices of the tetrahedron. Let O be the center of both the inscribed and circumscribed spheres. Let the inscribed sphere be tangent to the face ABC at the point E, and let its volume be V. Note that the radius of the inscribed sphere is OE and the radius of the circumscribed sphere is OD. Draw $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, and $\overline{OD}$ to obtain four congruent tetrahedra $ABCO, ABDO, ACDO$, and $BCDO$, each with volume 1/4 that of the original tetrahedron. Because the two tetrahedra $ABCD$ and $ABCO$ share the same base, $\triangle ABC$, the ratio of the distance from O to face ABC to the distance from D to face ABC is $1/4$; that is, $OD = 3 \cdot OE$. Thus the volume of the circumscribed sphere is $27V$. Extend $\overline{DE}$ to meet the circumscribed sphere at F. Then $DF = 2 \cdot DO = 6 \cdot OE$. Thus $EF = 2 \cdot OE$, so the sphere with diameter $\overline{EF}$ is congruent to the inscribed sphere, and thus has volume V. Similarly each of the other three spheres between the tetrahedron and the circumscribed sphere have volume V. The five congruent small spheres have no volume in common and lie entirely inside the circumscribed sphere, so the ratio $5V/27V$ is the probability that a point in the circumscribed sphere also lies in one of the small spheres. The fraction $5/27$ is closer to 0.2 than it is to any of the other choices.