UNIT 18 EXERCISES 21-25

COMPLEX

2009A

21. **Answer (C):** Because $x^{12}+ax^8+bx^4+c=p(x^4)$, the value of this polynomial is 0 if and only if

$$x^4 = 2009 + 9002\pi i \qquad \text{or} \qquad x^4 = 2009 \qquad \text{or} \qquad x^4 = 9002.$$

The first of these three equations has four distinct nonreal solutions, and the second and third each have two distinct nonreal solutions. Thus $p(x^4) = x^{12} + ax^8 + bx^4 + c$ has 8 distinct nonreal zeros.

2005B

22. **(E)** Note that

$$z_{n+1} = \frac{iz_n}{\overline{z_n}} = \frac{iz_n^2}{z_n\overline{z_n}} = \frac{iz_n^2}{|z_n|^2}.$$

Since $|z_0| = 1$, the sequence satisfies

$$z_1 = iz_0^2,\ z_2 = iz_1^2 = i\left(iz_0^2\right)^2 = -iz_0^4,$$

and, in general, when $k \geq 2$,

$$z_k = -iz_0^{2^k}.$$

Hence z_0 satisfies the equation $1 = -iz_0^{(2^{2005})}$, so $z_0^{(2^{2005})} = i$. Because every nonzero complex number has n distinct nth roots, this equation has 2^{2005} solutions. So there are 2^{2005} possible values for z_0.

OR

Define

$$\operatorname{cis}\theta = \cos\theta + i\sin\theta.$$

Then if $z_n = r \operatorname{cis}\theta$ we have

$$z_{n+1} = \frac{\operatorname{cis}(\theta + 90^\circ)}{\operatorname{cis}(-\theta)} = \operatorname{cis}(2\theta + 90^\circ).$$

The first terms of the sequence are $z_0 = \operatorname{cis}\alpha$, $z_1 = \operatorname{cis}(2\alpha + 90^\circ) = iz_0^2$, $z_2 = \operatorname{cis}(4\alpha + 270^\circ) = \operatorname{cis}(4\alpha - 90^\circ) = \frac{z_0^4}{i}$, $z_3 = \operatorname{cis}(8\alpha - 90^\circ) = \frac{z_0^8}{i}$, and, in general,

$$z_n = \frac{z_0^{(2^n)}}{i} \quad \text{for } n \geq 2.$$

So

$$z_{2005} = \frac{z_0^{(2^{2005})}}{i} = 1 \quad \text{and} \quad z_0^{(2^{2005})} = i.$$

As before, there are 2^{2005} possible solutions for z_0.

2018A

22. **Answer (A):** Let $z = a+bi$ be a solution of the first equation, where a and b are real numbers. Then $(a+bi)^2 = 4 + 4\sqrt{15}i$. Expanding the left-hand side and equating real and imaginary parts yields

$$a^2 - b^2 = 4 \quad \text{and} \quad 2ab = 4\sqrt{15}.$$

From the second equation, $b = \frac{2\sqrt{15}}{a}$, and substituting this into the first equation and simplifying gives $(a^2)^2 - 4a^2 - 60 = 0$, which factors as $(a^2-10)(a^2+6) = 0$. Because a is real, it follows that $a = \pm\sqrt{10}$, from which it then follows that $b = \pm\sqrt{6}$. Thus two vertices of the parallelogram are $\sqrt{10} + \sqrt{6}i$ and $-\sqrt{10} - \sqrt{6}i$. A similar calculation with the other given equation shows that the other two vertices of the parallelogram are $\sqrt{3} + i$ and $-\sqrt{3} - i$. The area of this parallogram can be computed using the shoelace formula, which gives the area of a polygon in terms of the coordinates of its vertices (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n) in clockwise or counter-clockwise order:

$$\frac{1}{2} \cdot \Big| (x_1y_2 + x_2y_3 + \cdots + x_{n-1}y_n + x_ny_1) - (y_1x_2 + y_2x_3 + \cdots + y_{n-1}x_n + y_nx_1) \Big|.$$

In this case $x_1 = \sqrt{10}$, $y_1 = \sqrt{6}$, $x_2 = \sqrt{3}$, $y_2 = 1$, $x_3 = -\sqrt{10}$, $y_3 = -\sqrt{6}$, $x_4 = -\sqrt{3}$, and $y_4 = -1$. The area is $6\sqrt{2} - 2\sqrt{10}$, and the requested sum of the four positive integers in this expression is 20.

OR

The solutions of $z^2 = 4 + 4\sqrt{15}i = 16\operatorname{cis} 2\theta_1$ are $z_1 = 4\operatorname{cis}\theta_1$ and its opposite, with $0 < \theta_1 < \frac{\pi}{4}$ and $\tan 2\theta_1 = \sqrt{15}$. Then $\cos 2\theta_1 = \frac{1}{4}$, and by the half-angle identities, $\cos\theta_1 = \frac{\sqrt{10}}{4}$ and $\sin\theta_1 = \frac{\sqrt{6}}{4}$. Similarly, the solutions of $z^2 = 2 + 2\sqrt{3}i = 4\operatorname{cis}\theta_2$ are $z_2 = 2\operatorname{cis}\theta_2$ and its opposite, with $0 < \theta_2 < \frac{\pi}{4}$ and $\tan 2\theta_2 = \sqrt{3}$. Then $\cos\theta_2 = \frac{\sqrt{3}}{2}$ and $\sin\theta_2 = \frac{1}{2}$.

The area of the parallelogram in the complex plane with vertices z_1, z_2, and their opposites is 4 times the area of the triangle with vertices 0, z_1, and z_2, and because the area of a triangle is one-half the product of the lengths of two of its sides and the sine of their included angle, it follows that the area of the parallelogram is

$$\begin{aligned} 4\left(\frac{1}{2} \cdot 4 \cdot 2 \cdot \sin(\theta_1 - \theta_2)\right) &= 16\left(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2\right) \\ &= 16\left(\frac{\sqrt{6}}{4} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{10}}{4} \cdot \frac{1}{2}\right) \\ &= 6\sqrt{2} - 2\sqrt{10}. \end{aligned}$$

Therefore, $p + q + r + s = 6 + 2 + 2 + 10 = 20$.

2001

23. **(A)** If r and s are the integer zeros, the polynomial can be written in the form

$$P(x) = (x-r)(x-s)(x^2+\alpha x+\beta).$$

The coefficient of x^3, $\alpha-(r+s)$, is an integer, so α is an integer. The coefficient of x^2, $\beta-\alpha(r+s)+rs$, is an integer, so β is also an integer. Applying the quadratic formula gives the remaining zeros as

$$\frac{1}{2}(-\alpha \pm \sqrt{\alpha^2-4\beta}) = -\frac{\alpha}{2} \pm i\frac{\sqrt{4\beta-\alpha^2}}{2}.$$

Answer choices (A), (B), (C), and (E) require that $\alpha=-1$, which implies that the imaginary parts of the remaining zeros have the form $\pm\sqrt{4\beta-1}/2$. This is true only for choice (A).

Note that choice (D) is not possible since this choice requires $\alpha=-2$, which produces an imaginary part of the form $\sqrt{\beta-1}$, which cannot be $\frac{1}{2}$.

2008A

23. **Answer (D):** Adding $1+i$ to each side of the given equation gives

$$1+i=(z^4+4z^3i-6z^2-4zi-i)+1+i=z^4+4z^3i-6z^2-4zi+1=(z+i)^4.$$

Let $w=z+i=r(\cos\theta+i\sin\theta)$. Since

$$i+1=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right),$$

the solutions of $w^4=1+i$ satisfy

$$r^4=\sqrt{2}\quad\text{and}\quad\theta=\frac{1}{4}\left(\frac{\pi}{4}+2k\pi\right)=\frac{\pi}{16}+\frac{\pi}{2}k,$$

for $k=0$, 1, 2, or 3. Thus

$$w_k=2^{1/8}\left(\cos\left(\frac{\pi}{16}+\frac{\pi}{2}k\right)+i\sin\left(\frac{\pi}{16}+\frac{\pi}{2}k\right)\right)\quad\text{for } k=0,1,2,\text{ or }3,$$

and the four solutions for $z=w-i$ are

$$z_k=2^{1/8}\left(\cos\left(\frac{\pi}{16}+\frac{\pi}{2}k\right)+i\sin\left(\frac{\pi}{16}+\frac{\pi}{2}k\right)\right)-i\quad\text{for } k=0,1,2,\text{ or }3.$$

Note that w_0, w_1, w_2, and w_3 are equally spaced around the circle of radius $2^{1/8}$ centered at $(0,0)$, so z_0, z_1, z_2, and z_3 are equally spaced around the circle of radius $2^{1/8}$ centered at $(0,-1)$. Therefore z_0, z_1, z_2, and z_3 are vertices of a square with side length $2^{1/8}\sqrt{2}=2^{5/8}$ and area $(2^{5/8})^2=2^{5/4}$.

OR

The Binomial Theorem gives

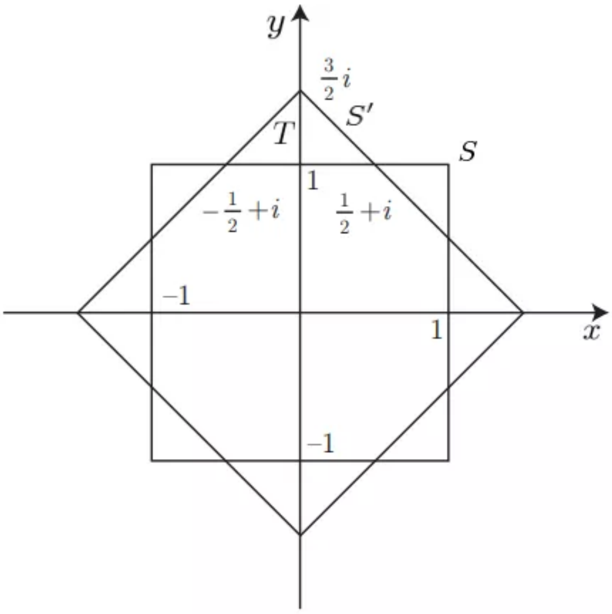
$$(z+i)^4=z^4+4z^3i-6z^2-4zi+1=(z^4+4z^3i-6z^2-4zi-i)+1+i=1+i.$$

Let a satisfy $a^4=1+i$, and let $w=z+i$. Then $w^4=a^4$, so the possible values for w are a, ia, $-a$, and $-ia$, which are the vertices of a square with diagonal $2|a|=2\sqrt[8]{2}$. The transformation $w=z+i$ is a translation, so it preserves area. Hence the area of the original polygon is $(2\sqrt[8]{2})^2/2=2\sqrt[4]{2}=2^{5/4}$.

2009B

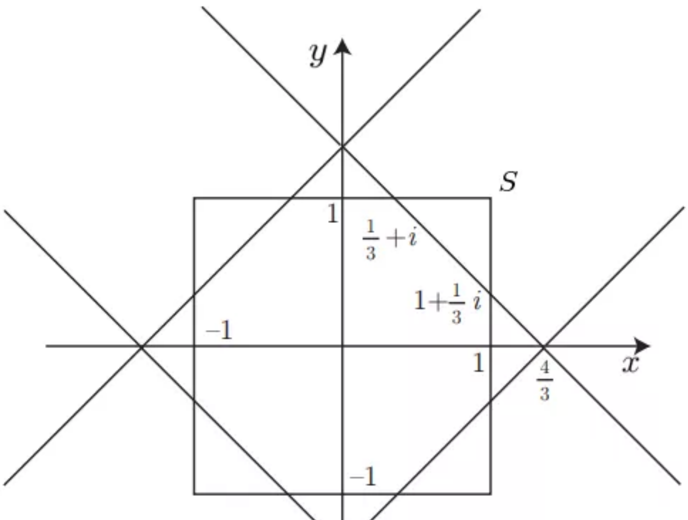
23. **Answer (D):** Let $f(z) = (\frac{3}{4} + \frac{3}{4}i)z$. The effect of multiplying z by $(\frac{3}{4} + \frac{3}{4}i)$ is to rotate z an angle equal to $\arg(\frac{3}{4} + \frac{3}{4}i) = \frac{\pi}{4}$ from the origin, and to magnify by a factor of $|\frac{3}{4} + \frac{3}{4}i| = \frac{3}{4}\sqrt{2}$. Thus the image S' of S under f is a square region with vertices $\pm\frac{3}{2}$ and $\pm\frac{3}{2}i$. The area of S' is $(\frac{3}{4}\sqrt{2} \cdot 2)^2$. The intersection of S and S' is an octagonal region obtained from S' by removing four congruent triangular regions. The topmost of these triangles T has vertices $\frac{1}{2} + i$, $\frac{3}{2}i$, and $-\frac{1}{2} + i$, so its area equals $\frac{1}{4}$. Then the requested probability is

$$\frac{(\frac{3}{4}\sqrt{2} \cdot 2)^2 - 4 \cdot \frac{1}{4}}{(\frac{3}{4}\sqrt{2} \cdot 2)^2} = \frac{7}{9}.$$



OR

The product is $(\frac{3}{4} + \frac{3}{4}i)(x + iy) = (\frac{3}{4}x - \frac{3}{4}y) + (\frac{3}{4}x + \frac{3}{4}y)i$. The point $x + iy$ will be in S if and only if $-1 \le \frac{3}{4}x - \frac{3}{4}y \le 1$ and $-1 \le \frac{3}{4}x + \frac{3}{4}y \le 1$, which are equivalent to $-\frac{4}{3} \le x - y \le \frac{4}{3}$ and $-\frac{4}{3} \le x + y \le \frac{4}{3}$. Thus $x + yi$ must be inside the square with vertices $\pm\frac{4}{3}$ and $\pm\frac{4}{3}i$. By symmetry we can look at just the first quadrant. Because the portion of S in the first quadrant has area 1, the desired probability is the area of the portion of the interior of this square within S. The squares intersect at $1 + \frac{1}{3}i$ and $\frac{1}{3} + i$, so the desired probability is $1 - \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{7}{9}$.

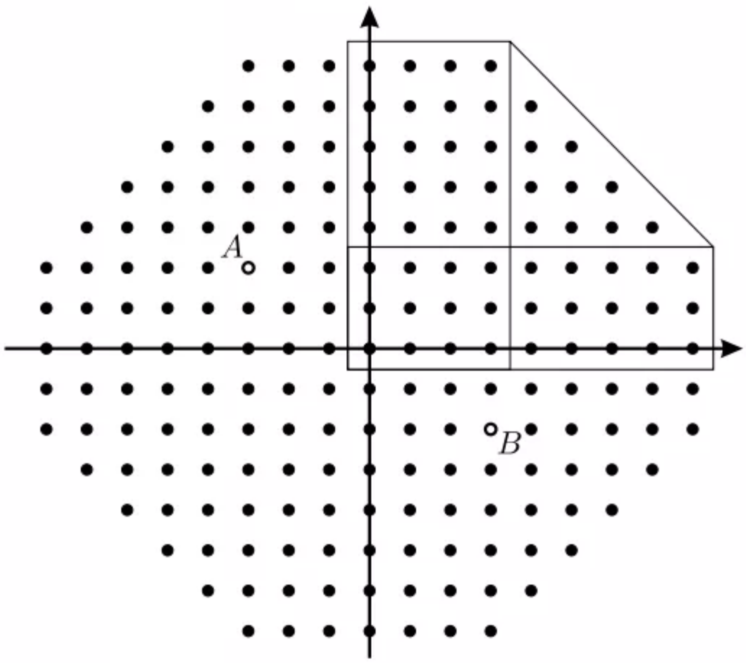


2011A

23. **Answer (C):** Let $X = (x, y)$. The distance traveled by the bug from A to X is at least $|x+3| + |y-2|$. Similarly, the distance traveled by the bug from X to B is at least $|x-3| + |y+2|$. It follows that X belongs to a path from A to B traveled by the bug if and only if

$$d = |x-3| + |x+3| + |y-2| + |y+2| \leq 20.$$

The expression for d is invariant if x is replaced by $-x$ or y is replaced by $-y$. By symmetry, it is enough to count the number of points X with $x \geq 0$ and $y \geq 0$, multiply by 4, and subtract the points that were overcounted, that is those in the x-axis or in the y-axis. Consider four cases:



Case 1. $0 \leq x \leq 3$ and $0 \leq y \leq 2$. In this case $|x-3| + |x+3| = 6$ and $|y-2| + |y+2| = 4$. Thus $d = 10 < 20$ and there are $4 \cdot 3 = 12$ points X in this case. This includes the origin and 5 other points for which $xy = 0$.

Case 2. $0 \leq x \leq 3$ and $y \geq 3$. In this case $|x-3|+|x+3| = 6$ and $|y-2|+|y+2| = 2y$. Thus $d = 6 + 2y \leq 20$ if and only if $y \leq 7$. There are $4 \cdot 5 = 20$ points X in this case. This includes 5 points for which $xy = 0$.

Case 3. $x \geq 4$ and $0 \leq y \leq 2$. In this case $|x-3| + |x+3| = 2x$ and $|y-2| + |y+2| = 4$. Thus $d = 4 + 2x \leq 20$ if and only if $x \leq 8$. There are $5 \cdot 3 = 15$ points X in this case. This includes 5 points for which $xy = 0$.

Case 4. $x \geq 4$ and $y \geq 3$. In this case $|x-3|+|x+3| = 2x$ and $|y-2|+|y+2| = 2y$. Thus $d = 2x + 2y \leq 20$ if and only if $x + y \leq 10$. The number of points X in this case is equal to

$$\sum_{x=4}^{7} \sum_{y=3}^{10-x} 1 = \sum_{x=4}^{7}(10 - x - 2) = \sum_{x=4}^{7}(8 - x) = 4 + 3 + 2 + 1 = 10,$$

and there are no points with $xy = 0$.

By symmetry the required total is $4(12 + 20 + 15 + 10) - 2(5 + 5 + 5) - 3 = 4 \cdot 57 - 2 \cdot 15 - 3 = 195$.

2012B

23. **Answer (B):** If z_0^k is equal to a positive real r, then $1 = |z_0|^k = |z_0^k| = |r| = r$, so $z_0^k = 1$. Suppose that $z_0^k = 1$. If $k = 1$, then $z_0 = 1$, but $P(1) = 4 + a + b + c + d \geq 4$ so $z_0 = 1$ is not a zero of the polynomial. If $k = 2$, then $z_0 = \pm 1$. If $z_0 = -1$, then $0 = P(-1) = (4 - a) + (b - c) + d$ and by assumption $4 \geq a$, $b \geq c$, and $d \geq 0$. Thus $a = 4$, $b = c$, and $d = 0$. Conversely, if $a = 4$, $b = c$, and $d = 0$, then $P(z) = 4z^4 + 4z^3 + bz^2 + bz = z(z + 1)(4z^2 + b)$ satisfies the required conditions. If $k = 3$, then $z_0 = 1$ or $z_0 = \gamma$ where γ is any of the roots of $\gamma^2 + \gamma + 1 = 0$. If $z_0 = \gamma$, then $0 = P(\gamma) = 4\gamma + a + b(-1 - \gamma) + c\gamma + d = (a - b) + d + \gamma((4 - b) + c)$ and by assumption $a \geq b$, $d \geq 0$, $4 \geq b$, and $c \geq 0$. Thus $a = b$, $d = 0$, $b = 4$, and $c = 0$. Conversely, if $a = b = 4$ and $c = d = 0$, then $P(z) = 4z^4 + 4z^3 + 4z^2 = 4z^2(z^2 + z + 1)$ satisfies the given conditions because $z_0 = \cos(2\pi/3) + i\sin(2\pi/3)$ is a zero of this polynomial. If $k = 4$, then $z_0 = \pm 1$ or $z_0 = \pm i$. If $z_0 = \pm i$, then $0 = P(\pm i) = 4 \mp ia - b \pm ic + d = (4 - b) + d \mp i(a - c)$ and by assumption $4 \geq b$, $d \geq 0$, and $4 \geq a \geq b \geq c$. Thus $b = 4$, $d = 0$, and $a = c = 4$. Conversely, if $a = b = c = 4$ and $d = 0$, then $P(z) = 4z^4 + 4z^3 + 4z^2 + 4z = 4z(z + 1)(z^2 + 1)$ satisfies the given conditions, but it was already considered in the case when $z_0 = -1$. The remaining case is that z_0^k is not a positive real number for $1 \leq k \leq 4$. In this case,

$$4z^5 - (z - 1)P(z) = z^4(4 - a) + z^3(a - b) + z^2(b - c) + z(c - d) + d.$$

If $z = z_0$, then the triangle inequality yields

$$\begin{aligned} 4 &= |z_0^4(4 - a) + z_0^3(a - b) + z_0^2(b - c) + z_0(c - d) + d| \\ &\leq \left|z_0^4(4 - a)\right| + \left|z_0^3(a - b)\right| + \left|z_0^2(b - c)\right| + |z_0(c - d)| + |d| \\ &= |z_0|^4(4 - a) + |z_0|^3(a - b) + |z_0|^2(b - c) + |z_0|(c - d) + d \\ &= 4 - a + a - b + b - c + c - d + d = 4. \end{aligned}$$

Thus equality must occur throughout. This means that the vectors $v_4 = z_0^4(4 - a)$, $v_3 = z_0^3(a - b)$, $v_2 = z_0^2(b - c)$, $v_1 = z_0(c - d)$, and $v_0 = d$ are parallel and they belong to the same quadrant. If two of these vectors are nonzero, then the quotient must be a positive real number; but dividing the vector with the largest exponent of z_0 by the other would yield a positive rational number times z_0^k for some $1 \leq k \leq 4$. Because not all of the v_j can be zero, it follows that there is exactly one of them that is nonzero. If $v_0 = d \neq 0$ and $v_1 = v_2 = v_3 = v_4 = 0$, then $4 = a = b = c = d$, and $P(z) = 4z^4 + 4z^3 + 4z^2 + 4z + 4$ satisfies the given conditions because $z_0 = \cos(2\pi/5) + i\sin(2\pi/5)$ is a zero of this polynomial. Finally, if $v_j \neq 0$ for some $1 \leq j \leq 4$ and the rest are zero, then $4z_0^5 = v_j = z_0^j n$ for some positive integer n, and so $z_0^{5-j} = \frac{1}{4}n$ is a positive real.

Therefore the complete list of polynomials is: $4z^4 + 4z^3 + 4z^2 + 4z + 4$, $4z^4 + 4z^3 + 4z^2$, and $4z^4 + 4z^3 + bz^2 + bz$ with $0 \leq b \leq 4$. The required sum is $20 + 12 + \sum_{b=0}^{4}(8 + 2b) = 32 + 40 + (2 + 4 + 6 + 8) = 92$.

2002A 24. **(E)** Let $z = a + bi$, $\overline{z} = a - bi$, and $|z| = \sqrt{a^2 + b^2}$. The given relation becomes $z^{2002} = \overline{z}$. Note that

$$|z|^{2002} = \left|z^{2002}\right| = |\overline{z}| = |z|,$$

from which it follows that

$$|z|\left(|z|^{2001} - 1\right) = 0.$$

Hence $|z| = 0$, and $(a, b) = (0, 0)$, or $|z| = 1$. In the case $|z| = 1$, we have $z^{2002} = \overline{z}$, which is equivalent to $z^{2003} = \overline{z} \cdot z = |z|^2 = 1$. Since the equation $z^{2003} = 1$ has 2003 distinct solutions, there are altogether $1 + 2003 = 2004$ ordered pairs that meet the required conditions.

2011B

24. **Answer (B):** Factoring or using the quadratic formula with z^4 as the variable yields $P(z) = (z^4 - 1)(z^4 + (4\sqrt{3} + 7))$. Moreover, $4\sqrt{3} + 7 = (\sqrt{3} + 2)^2$ and $2(\sqrt{3}+2) = 2\sqrt{3}+4 = (\sqrt{3}+1)^2$; thus $4\sqrt{3}+7 = (\frac{1}{2}(\sqrt{6}+\sqrt{2}))^4$. If $w = \frac{1}{2}(\sqrt{3}+1)$, then the eight zeros of $P(z)$ are $1, -1, i, -i, w(1+i), w(-1+i), w(-1-i)$, and $w(1-i)$.

The distances from 1 to the other zeros are

$$|1-(-1)| = 2, |1 \pm i| = \sqrt{2}, |1 - w(1 \pm i)| = \sqrt{(1-w)^2 + w^2} = \sqrt{2}, \text{ and}$$
$$|1 - w(-1 \pm i)| = \sqrt{(1+w)^2 + w^2} = \sqrt{2\sqrt{3}+4} = \sqrt{3}+1.$$

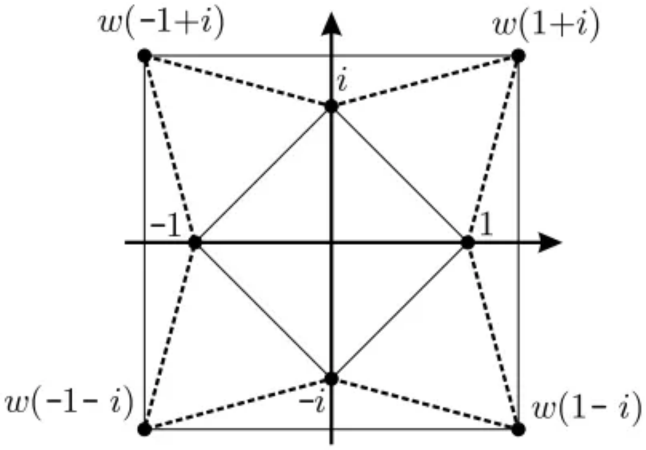
Similarly, the distances from $w(1+i)$ to the other zeros are

$$|w(1+i) - w(1-i)| = |w(1+i) - w(-1+i)| = 2w = \sqrt{3}+1,$$
$$|w(1+i) - w(-1-i)| = 2\sqrt{2}w = \sqrt{6}+\sqrt{2},$$

and by symmetry,

$$|w(1+i) - 1| = |w(1+i) - i| = \sqrt{2}, \text{ and}$$
$$|w(1+i) + 1| = |w(1+i) + i| = \sqrt{3}+1.$$

Because the set of zeros is 4-fold symmetric with respect to the origin, it follows that every line segment joining two of the zeros has length at least $\sqrt{2}$. This shows that any polygon with vertices at the zeros has perimeter at least $8\sqrt{2}$. Finally, note that the polygon with consecutive vertices $1, w(1+i), i, w(-1+i), -1, w(-1-i), -i$, and $w(1-i)$ has perimeter $8\sqrt{2}$.



2013A 25. **Answer (A):** Let $H = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$. If $z_1, z_2 \in H$ and $f(z_1) = f(z_2)$, then $z_1^2 - z_2^2 + i(z_1 - z_2) = (z_1 - z_2)(z_1 + z_2 + i) = 0$. Because $\text{Im}(z_1) > 0$ and $\text{Im}(z_2) > 0$, it follows that $z_1 + z_2 + i \neq 0$. Thus $z_1 = z_2$; that is, the function f is one-to-one on H. Let r be a positive real number. Note that $f(r) = r^2 + 1 + ir$ describes the top part of the parabola $x = y^2 + 1$. Similarly, $f(-r) = r^2 + 1 - ir$ describes the bottom part of the parabola $x = y^2 + 1$. Because $f(i) = -1$, it follows that the image set $f(H)$ equals $\{w \in \mathbb{C} : \text{Re}(w) < (\text{Im}(w))^2 + 1\}$. Thus the set of complex numbers $w \in f(H)$ with integer real and imaginary parts of absolute value at most 10 is equal to

$$S = \{w = a + ib \in \mathbb{C} : a, b \in \mathbb{Z},\ |a| \leq 10,\ |b| \leq 10, \text{ and } a < b^2 + 1\}.$$

Because f is one-to-one, the required answer is $|f^{-1}(S)| = |S|$ and

$$|S| = 21^2 - \sum_{b=-3}^{3} \sum_{a=b^2+1}^{10} 1 = 441 - \sum_{b=-3}^{3} (10 - b^2)$$
$$= 441 - (1 + 6 + 9 + 10 + 9 + 6 + 1) = 399.$$

2017A

25. **Answer (E):** If z_j is an element of the set $A = \{\sqrt{2}i, -\sqrt{2}i\}$, then $|z_j| = \sqrt{2}$. Otherwise z_j is an element of

$$B = V \setminus A = \left\{ \frac{1}{\sqrt{8}}(1+i), \frac{1}{\sqrt{8}}(-1+i), \frac{1}{\sqrt{8}}(1-i), \frac{1}{\sqrt{8}}(-1-i) \right\}$$

and $|z_j| = \frac{1}{2}$. It follows that $|P| = \prod_{j=1}^{12} |z_j| = 1$ exactly when 8 of the 12 factors z_j are in A and 4 of the factors are in B. The product of 8 complex numbers each of which is in A is a real number, either 16 or -16. The product of 4 numbers each of which is in B is one of $\frac{1}{16}$, $\frac{1}{16}i$, $-\frac{1}{16}$, or $-\frac{1}{16}i$. Thus a product $P = \prod_{j=1}^{12} z_j$ is -1 exactly when 8 of the z_j are from A, 4 of the z_j are from B, and the last of the 4 elements from B is chosen so that the product is -1 rather than i, $-i$, or 1. Because the probability is $\frac{1}{3}$ that a particular factor z_j is from A, the probability is $\frac{2}{3}$ that a particular factor z_j is from B, and the probability is $\frac{1}{6}$ that a particular factor z_j is a specific element of V, the probability that the product P will be -1 is given by

$$\binom{12}{4}\left(\frac{1}{3}\right)^8\left(\frac{2}{3}\right)^3\left(\frac{1}{6}\right) = \frac{12 \cdot 11 \cdot 10 \cdot 9}{4 \cdot 3 \cdot 2 \cdot 1} \cdot \frac{1}{3^8} \cdot \frac{2^3}{3^3} \cdot \frac{1}{6} = \frac{2^2 \cdot 5 \cdot 11}{3^{10}}.$$