UNIT 17 EXERCISES 21-25

TRIG

2004A 21. **(D)** The given series is geometric with an initial term of 1 and a common ratio of $\cos^2\theta$, so its sum is

$$5 = \sum_{n=0}^{\infty} \cos^{2n}\theta = \frac{1}{1-\cos^2\theta} = \frac{1}{\sin^2\theta}.$$

Therefore $\sin^2\theta = \frac{1}{5}$, and

$$\cos 2\theta = 1 - 2\sin^2\theta = 1 - \frac{2}{5} = \frac{3}{5}.$$

2003B

23. (A) The intercepts occur where $\sin(1/x) = 0$, that is, where $x = 1/(k\pi)$ and k is a nonzero integer. Solving

$$0.0001 < \frac{1}{k\pi} < 0.001$$

yields

$$\frac{1000}{\pi} < k < \frac{10,000}{\pi}.$$

Thus the number of x intercepts in $(0.0001, 0.001)$ is

$$\left\lfloor \frac{10,000}{\pi} \right\rfloor - \left\lfloor \frac{1000}{\pi} \right\rfloor = 3183 - 318 = 2865,$$

which is closest to 2900.

2006B

23. **(E)** Let D, E, and F be the reflections of P about $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, respectively. Then $\angle FAD = \angle DBE = 90°$, and $\angle ECF = 180°$. Thus the area of pentagon $ADBEF$ is twice that of $\triangle ABC$, so it is s^2.

2007A 24. **Answer (D):** Note that $F(n)$ is the number of points at which the graphs of $y = \sin x$ and $y = \sin nx$ intersect on $[0, \pi]$. For each n, $\sin nx \geq 0$ on each interval $[(2k-2)\pi/n, (2k-1)\pi/n]$ where k is a positive integer and $2k - 1 \leq n$. The number of such intervals is $n/2$ if n is even and $(n+1)/2$ if n is odd. The graphs intersect twice on each interval unless $\sin x = 1 = \sin nx$ at some point in the interval, in which case the graphs intersect once. This last equation is satisfied if and only if $n \equiv 1 \pmod 4$ and the interval contains $\pi/2$. If n is even, this count does not include the point of intersection at $(\pi, 0)$. Therefore $F(n) = 2(n/2) + 1 = n + 1$ if n is even, $F(n) = 2(n+1)/2 = n + 1$ if $n \equiv 3 \pmod 4$, and $F(n) = n$ if $n \equiv 1 \pmod 4$. Hence

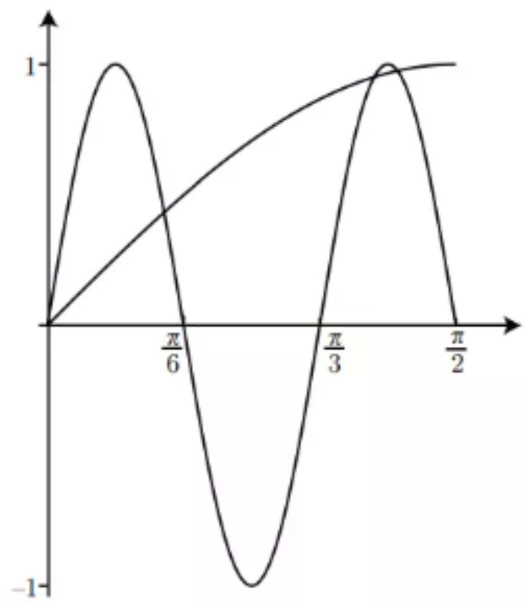
$$\sum_{n=2}^{2007} F(n) = \left(\sum_{n=2}^{2007} (n+1)\right) - \left\lfloor \frac{2007-1}{4} \right\rfloor = \frac{(2006)(3+2008)}{2} - 501 = 2{,}016{,}532.$$

2009B

24. **Answer (B):** Let $f(x) = \sin^{-1}(\sin 6x)$ and $g(x) = \cos^{-1}(\cos x)$. If $0 \le x \le \pi$, then $g(x) = x$. If $0 \le x \le \pi/12$, then $f(x) = 6x$. Note also that $\sin\left(6\left(\frac{\pi}{6} - x\right)\right) = \sin 6x$, $\sin\left(6\left(\frac{\pi}{3} - x\right)\right) = -\sin 6x$, and $\sin\left(6\left(\frac{\pi}{3} + x\right)\right) = \sin 6x$, from which it follows that $f(\frac{\pi}{6} - x) = f(x)$, $f(\frac{\pi}{3} - x) = -f(x)$, and $f(\frac{\pi}{3} + x) = f(x)$. Thus the graph of $y = f(x)$ has period $\frac{\pi}{3}$ and consists of line segments with slopes of 6 or -6 and endpoints at $((4k+1)\frac{\pi}{12}, \frac{\pi}{2})$ and $((4k+3)\frac{\pi}{12}, -\frac{\pi}{2})$ for integer values of k. The graphs of f and g intersect twice in the interval $[0, \frac{\pi}{6}]$ and twice more in the interval $[\frac{\pi}{3}, \frac{\pi}{2}]$. If $\frac{\pi}{2} < x \le \pi$, then $g(x) = x > \frac{\pi}{2}$, so the graphs of f and g do not intersect.

OR

In the range $[0, \pi]$, we have $\cos^{-1}(\cos x) = x$. Since the range of $\sin^{-1} x$ is $[-\frac{\pi}{2}, \frac{\pi}{2}]$, it suffices to solve the equation $\sin^{-1}(\sin(6x)) = x$ on the interval $[0, \frac{\pi}{2}]$. Since $\sin x$ is one-to-one in $[0, \frac{\pi}{2}]$, we can consider the equivalent equation $\sin(\sin^{-1}(\sin(6x))) = \sin x$, or $\sin(6x) = \sin x$. Let $f(x) = \sin(6x)$ and $g(x) = \sin x$. Note that $f(0) = 0$, $f(\frac{\pi}{12}) = 1$, $f(\frac{\pi}{4}) = -1$, $f(\frac{5\pi}{12}) = 1$, and $f(\frac{\pi}{2}) = 0$. Moreover $f(x)$ is increasing on $(0, \frac{\pi}{12})$ and $(\frac{\pi}{4}, \frac{5\pi}{12})$, and decreasing on $(\frac{\pi}{12}, \frac{\pi}{4})$ and $(\frac{5\pi}{12}, \frac{\pi}{2})$. Similarly $g(0) = 0$, $g(\frac{\pi}{2}) = 1$, and $g(x)$ is increasing on $[0, \frac{\pi}{2}]$. Thus the graphs of $y = f(x)$ and $y = g(x)$ intersect at $x = 0$, once in the interval $[\frac{\pi}{12}, \frac{\pi}{4}]$, once in the interval $[\frac{\pi}{4}, \frac{5\pi}{12}]$, and once more in the interval $[\frac{5\pi}{12}, \frac{\pi}{2}]$. Therefore there are 4 solutions to the given equation.



2010A 24. **Answer (B):**

Let $g(x) = \sin(\pi x) \cdot \sin(2\pi x) \cdot \sin(3\pi x) \cdots \sin(8\pi x)$. The domain of $f(x)$ is the union of all intervals on which $g(x) > 0$. Note that $\sin(n\pi(1x)) = (-1)^{k+1}\sin(n\pi x)$, so $g(1x) = g(x)$. Because $g(1/2) = 0$, it suffices to consider the subintervals of $(0, 1/2)$ on which $g(x) > 0$. In this interval the distinct solutions of the equation $g(x) = 0$ are the numbers k/n, where $2 \le n \le 8$, $1 \le k < n/2$, and k and n are relatively prime. For $n = 2, 3, 4, 5, 6, 7,$ and 8 there are, respectively, 0, 1, 1, 2, 1, 3, and 2 values of k. Thus there are $1 + 1 + 2 + 1 + 3 + 2 = 10$ solutions of $g(x) = 0$ in the interval $(0, 1/2)$. The sign of $g(x)$ changes at k/n unless an even number of factors of $g(x)$ are zero at k/n, that is unless there are an even number of ways to represent k/n as a rational number with a positive denominator not exceeding 8. Thus the sign of $g(n)$ changes except at $1/4 = 2/8$ and $1/3 = 2/6$.

Let the solutions of $g(x) = 0$ in the interval $(0, 1/2)$ be $x_1, x_2, \ldots, x_{10}$ in increasing order, and let $x_0 = 0$ and $x_{11} = 1/2$. It is easily verified that $x_5 = 1/4$ and $x_7 = 1/3$, so for $0 \le j \le 10$, the sign of $g(x)$ changes at x_j except for $j = 5$ and 7. Because 5 and 7 have the same parity and $g(x) > 0$ in (x_0, x_1), the solution of $g(x) > 0$ in $(0, 1/2)$ consists of 6 disjoint open intervals. The solution of $g(x) > 0$ in $(1/2, 1)$ also consists of 6 disjoint open intervals, so the requested number of intervals is 12.

2015A

24. **Answer (D):** There are 20 possible values for each of a and b, namely those in the set

$$S = \left\{0, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \frac{9}{5}\right\}.$$

If x and y are real numbers, then $(x+iy)^2 = x^2 - y^2 + i(2xy)$ is real if and only if $xy = 0$, that is, $x = 0$ or $y = 0$. Therefore $(x+iy)^4$ is real if and only if $x^2 - y^2 = 0$ or $xy = 0$, that is, $x = 0$, $y = 0$, or $x = \pm y$. Thus $((\cos(a\pi) + i\sin(b\pi))^4$ is a real number if and only if $\cos(a\pi) = 0$, $\sin(b\pi) = 0$, or $\cos(a\pi) = \pm\sin(b\pi)$. If $\cos(a\pi) = 0$ and $a \in S$, then $a = \frac{1}{2}$ or $a = \frac{3}{2}$ and b has no restrictions, so there are 40 pairs (a, b) that satisfy the condition. If $\sin(b\pi) = 0$ and $b \in S$, then $b = 0$ or $b = 1$ and a has no restrictions, so there are 40 pairs (a, b) that satisfy the condition, but there are 4 pairs that have been counted already, namely $(\frac{1}{2}, 0)$, $(\frac{1}{2}, 1)$, $(\frac{3}{2}, 0)$, and $(\frac{3}{2}, 1)$. Thus the total so far is $40 + 40 - 4 = 76$.

Note that $\cos(a\pi) = \sin(b\pi)$ implies that $\cos(a\pi) = \cos(\pi(\frac{1}{2} - b))$ and thus $a \equiv \frac{1}{2} - b \pmod 2$ or $a \equiv -\frac{1}{2} + b \pmod 2$. If the denominator of $b \in S$ is 3 or 5, then the denominator of a in simplified form would be 6 or 10, and so $a \notin S$. If $b = \frac{1}{2}$ or $b = \frac{3}{2}$, then there is a unique solution to either of the two congruences, namely $a = 0$ and $a = 1$, respectively. For every $b \in \{\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}\}$, there is exactly one solution $a \in S$ to each of the previous congruences. None of the solutions are equal to each other because if $\frac{1}{2} - b \equiv -\frac{1}{2} + b \pmod 2$, then $2b \equiv 1 \pmod 2$; that is, $b = \frac{1}{2}$ or $b = \frac{3}{2}$. Similarly, $\cos(a\pi) = -\sin(b\pi) = \sin(-b\pi)$ implies that $\cos(a\pi) = \cos(\pi(\frac{1}{2} + b))$ and thus $a \equiv \frac{1}{2} + b \pmod 2$ or $a \equiv -\frac{1}{2} - b \pmod 2$. If the denominator of $b \in S$ is 3 or 5, then the denominator of a would be 6 or 10, and so $a \notin S$. If $b = \frac{1}{2}$ or $b = \frac{3}{2}$, then there is a unique solution to either of the two congruences, namely $a = 1$ and $a = 0$, respectively. For every $b \in \{\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}\}$, there is exactly one solution $a \in S$ to each of the previous congruences, and, as before, none of these solutions are equal to each other. Thus there are a total of $2 + 8 + 2 + 8 = 20$ pairs $(a, b) \in S^2$ such that $\cos(a\pi) = \pm\sin(b\pi)$. The requested probability is $\frac{76+20}{400} = \frac{96}{400} = \frac{6}{25}$.

Note: By de Moivre's Theorem the fourth power of the complex number $x + iy$ is real if and only if it lies on one of the four lines $x = 0$, $y = 0$, $x = y$, or $x = -y$. Then the counting of (a, b) pairs proceeds as above.

2014B 25. **Answer (D):** If $x = \frac{1}{2}\pi y$, then the given equation is equivalent to

$$2\cos(\pi y)\left(\cos(\pi y) - \cos\left(\frac{4028\pi}{y}\right)\right) = \cos(2\pi y) - 1.$$

Dividing both sides by 2 and using the identity $\frac{1}{2}(1-\cos(2\pi y)) = \sin^2(\pi y)$ yields

$$\cos^2(\pi y) - \cos(\pi y)\cos\left(\frac{4028\pi}{y}\right) = \frac{1}{2}(\cos(2\pi y) - 1) = -\sin^2(\pi y).$$

This is equivalent to

$$1 = \cos(\pi y)\cos\left(\frac{4028\pi}{y}\right).$$

Thus either $\cos(\pi y) = \cos(\frac{4028\pi}{y}) = 1$ or $\cos(\pi y) = \cos(\frac{4028\pi}{y}) = -1$. It follows that y and $\frac{4028}{y}$ are both integers having the same parity. Therefore y cannot be odd or a multiple of 4. Finally, let $y = 2a$ with a a positive odd divisor of $4028 = 2^2 \cdot 19 \cdot 53$, that is $a \in \{1, 19, 53, 19 \cdot 53\}$. Then $\cos(\pi y) = \cos(2a\pi) = 1$ and $\cos(\frac{4028\pi}{y}) = \cos(\frac{2014\pi}{a}) = 1$. Therefore the sum of all solutions x is $\pi(1 + 19 + 53 + 19 \cdot 53) = \pi(19+1)(53+1) = 1080\pi$.

1999 27. **(A)** Square both sides of the equations and add the results to obtain

$$9(\sin^2 A + \cos^2 A) + 16(\sin^2 B + \cos^2 B) + 24(\sin A \cos B + \sin B \cos A) = 37.$$

Hence, $24\sin(A+B) = 12$. Thus $\sin C = \sin(180° - A - B) = \sin(A+B) = \frac{1}{2}$, so $\angle C = 30°$ or $\angle C = 150°$. The latter is impossible because it would imply that $A < 30°$ and consequently that $3\sin A + 4\cos B < 3 \cdot \frac{1}{2} + 4 < 6$, a contradiction. Therefore $\angle C = 30°$.

Challenge. Prove that there is a unique such triangle (up to similarity), the one for which $\cos A = \frac{5-12\sqrt{3}}{37}$ and $\cos B = \frac{66-3\sqrt{3}}{74}$.