UNIT 16 EXERCISES 21-25

FUNCTIONS

2011A 21. **Answer (D):** Let the arithmetic and geometric means of x and y be $10a+b$ and $10b+a$, respectively. Then

$$\frac{x+y}{2} = 10a+b \Rightarrow (x+y)^2 = 400a^2+80ab+4b^2$$

and

$$\sqrt{xy} = 10b+a \Rightarrow xy = 100b^2+20ab+a^2,$$

so

$$(x-y)^2 = (x+y)^2 - 4xy = 396(a^2-b^2) = 11\cdot 6^2\cdot (a+b)(a-b)$$

Because x and y are distinct, a and b are distinct digits, and the last expression is a perfect square if and only if $a+b=11$ and $a-b$ is a perfect square. The cases $a-b=1$, 4, and 9 give solutions $(a,b)=(6,5)$, $(7.5,3.5)$, and $(10,1)$, respectively. Because a and b are digits only the first solution is valid. Thus $(x-y)^2 = 11\cdot 6^2\cdot 11 = 66^2$ and $|x-y|=66$. Note that the given conditions are satisfied if $\{x,y\}=\{32,98\}$.

2007A 22. **Answer (D):** If $n \le 2007$, then $S(n) \le S(1999) = 28$. If $n \le 28$, then $S(n) \le S(28) = 10$. Therefore if n satisfies the required condition it must also satisfy

$$n \ge 2007 - 28 - 10 = 1969.$$

In addition, $n, S(n),$ and $S(S(n))$ all leave the same remainder when divided by 9. Because 2007 is a multiple of 9, it follows that $n, S(n),$ and $S(S(n))$ must all be multiples of 3. The required condition is satisfied by 4 multiples of 3 between 1969 and 2007, namely 1977, 1980, 1983, and 2001.

Note: There appear to be many cases to check, that is, all the multiples of 3 between 1969 and 2007. However, for $1987 \le n \le 1999$, we have $n + S(n) \ge 1990 + 19 = 2009$, so these numbers are eliminated. Thus we need only check 1971, 1974, 1977, 1980, 1983, 1986, 2001, and 2004.

2009A

23. **Answer (D):** Let (h, k) be the vertex of the graph of f. Because the graph of f intersects the x-axis twice, we can assume that $f(x) = a(x-h)^2 + k$ with $\frac{-k}{a} > 0$. Let $s = \sqrt{\frac{-k}{a}}$; then the x-intercepts of the graph of f are $h \pm s$. Because $g(x) = -f(100-x) = -a(100-x-h)^2 - k$, it follows that the x-intercepts of the graph of g are $100 - h \pm s$.

The graph of g contains the point (h, k); thus

$$k = f(h) = g(h) = -a(100-2h)^2 - k,$$

from which $h = 50 \pm \frac{\sqrt{2}}{2}s$. Regardless of the sign in the expression for h, the four x-intercepts in order are

$$50 - s\left(1 + \frac{\sqrt{2}}{2}\right) < 50 - s\left(1 - \frac{\sqrt{2}}{2}\right) < 50 + s\left(1 - \frac{\sqrt{2}}{2}\right) < 50 + s\left(1 + \frac{\sqrt{2}}{2}\right).$$

Because $x_3 - x_2 = 150$, it follows that $150 = s(2-\sqrt{2})$, that is $s = 150\left(1 + \frac{\sqrt{2}}{2}\right)$. Therefore $x_4 - x_1 = s(2+\sqrt{2}) = 450 + 300\sqrt{2}$, and then $m + n + p = 450 + 300 + 2 = 752$.

OR

The graphs of f and g intersect the x-axis twice each. By symmetry, and because the graph of g contains the vertex of f, we can assume x_1 and x_3 are the roots of f, and x_2 and x_4 are the roots of g. A point (p, q) is on the graph of f if and only if $(100-p, -q)$ is on the graph of g, so the two graphs are reflections of each other with respect to the point $(50, 0)$. Thus $x_2 + x_3 = x_1 + x_4 = 100$, and since $x_3 - x_2 = 150$, it follows that $x_2 = -25$ and $x_3 = 125$. The average of x_1 and $x_3 = 125$ is h. It follows that $x_1 = 2h - 125$, from which $x_4 = 100 - x_1 = 225 - 2h$, and $x_4 - x_1 = 350 - 4h$.

Moreover, $f(x) = a(x - x_1)(x - x_3) = a(x + 125 - 2h)(x - 125)$ and $g(x) = -f(100-x) = -a(x+25)(x+2h-225)$. The vertex of the graph of f lies on the graph of g; thus

$$1 = \frac{f(h)}{g(h)} = \frac{(125-h)(h-125)}{-(h+25)(3h-225)},$$

from which $h = -25 \pm 75\sqrt{2}$. However, $h < x_2 < 0$; thus $h = -25 - 75\sqrt{2}$. Therefore $x_4 - x_1 = 450 + 300\sqrt{2}$ and then $m + n + p = 450 + 300 + 2 = 752$

2009A

24. **Answer (E):** Define the *k-iterated logarithm* as follows: $\log_2^1 x = \log_2 x$ and $\log_2^{k+1} x = \log_2(\log_2^k x)$ for $k \geq 1$. Because $\log_2 T(n+1) = T(n)$ for $n \geq 1$, it follows that $\log_2 A = T(2009)\log_2 T(2009) = T(2009)T(2008)$ and $\log_2 B = A\log_2 T(2009) = A \cdot T(2008)$. Then $\log_2^2 B = \log_2 A + \log_2 T(2008) = T(2009)T(2008) + T(2007)$. Now,

$$\log_2^3 B > \log_2(T(2009)T(2008)) > \log_2 T(2009) = T(2008),$$

and recursively for $k \geq 1$,

$$\log_2^{k+3} B > T(2008-k).$$

In particular $\log_2^{2010} B > T(1) = 2$, and then $\log_2^{2012} B > 0$. Thus $\log_2^{2013} B$ is defined.

On the other hand, because $T(2007) < T(2008)T(2009)$ and $1 + T(2007) < T(2008)$, it follows that

$$\log_2^3 B < \log_2\left(2T(2008)T(2009)\right) = 1 + T(2007) + T(2008) < 2T(2008) \text{ and}$$
$$\log_2^4 B < \log_2\left(2T(2008)\right) = 1 + T(2007) < T(2008).$$

Applying $\log_2$ recursively for $k \geq 1$ we get

$$\log_2^{4+k} B < T(2008-k).$$

In particular $\log_2^{2011} B < T(1) = 2$, and then $\log_2^{2013} B < 0$. Thus $\log_2^{2014} B$ is undefined.

2012B

24. **Answer (D):** Let $S_N = (f_1(N), f_2(N), f_3(N), \ldots)$. If N_1 divides N_2, then $f_1(N_1)$ divides $f_1(N_2)$. Thus S_{N_2} is unbounded if S_{N_1} is unbounded. Call N *essential* if S_N is unbounded and $N \le 400$ is not divisible by any smaller number n such that S_n is unbounded. Assume $N = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ is essential. If $e_j = 1$ for some j, then $f_1(N) = f_1(\frac{N}{p_j})$. Let $n = \frac{N}{p_j}$ and note that S_N and S_n coincide after the first term and consequently S_n is unbounded. This contradicts the fact that N is essential. Thus $e_j \ge 2$ for all $1 \le j \le k$. Moreover, $(p_1 p_2 \cdots p_k)^2 \le p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k} = N \le 400$; thus $p_1 p_2 \cdots p_k \le \sqrt{400} = 20$. Because $2 \cdot 3 \cdot 5 > 20$ it follows that $k \le 2$.

First analyze the case when $n = 2^a \cdot 3^b$. In that case $f_2(n) = f_1(2^{2b-2} \cdot 3^{a-1}) = 2^{2a-4} \cdot 3^{2b-3}$; thus S_n is unbounded if and only if $a \ge 5$ or $b \ge 4$, and n is essential if and only if $n = 2^5$ or $n = 3^4$.

If $k = 1$, then $N = p^e$ for some prime $p \le 19$. The cases $p = 2$ or $p = 3$ have been considered before. If $p = 5$, then $f_1(5^a) = 2^{a-1} \cdot 3^{a-1}$ and because $a \le 3$, no power of 5 in the given range is essential. If $p = 7$, then $f_1(7^a) = 2^{3a-3}$, and thus $N = 7^3$ is essential. If $p \ge 11$, then $p^3 > 400$. Because $f_1(11^2) = 2^2 \cdot 3$, $f_2(13^2) = f_1(2 \cdot 7) = 1$, $f_1(17^2) = 2 \cdot 3^2$, and $f_2(19^2) = f_1(2^2 \cdot 5) = 3$, no powers of 11, 13, 17, or 19 are essential.

If $k = 2$, then the only possible pairs of primes (p_1, p_2) are $(2,3)$, $(2,5)$, $(2,7)$, and $(3,5)$. The pair $(2,3)$ was analyzed before and it yields no essential N. If $N = 2^a \cdot 5^b \le 400$ is essential, then $2 \le a \le 4$ and $b = 2$. Moreover $f_1(N) = 2 \cdot 3^a$, so $a = 4$ and thus only $N = 2^4 \cdot 5^2$ is essential in this case. If $(p_1, p_2) = (2,7)$ or $(3,5)$ and $N = p_1^{e_1} p_2^{e_2} \le 400$ is essential, then $N \in \{2^2 \cdot 7^2, 2^3 \cdot 7^2, 3^2 \cdot 5^2\}$. Because $f_1(2^2 \cdot 7^2) = 2^3 \cdot 3$, $f_1(2^3 \cdot 7^2) = 2^3 \cdot 3^2$, and $f_1(3^2 \cdot 5^2) = 2^3 \cdot 3$, it follows that there are no essential N in this case.

Therefore the only essential values of N are $2^5 = 32$, $3^4 = 81$, $7^3 = 343$, and $2^4 \cdot 5^2 = 400$. These values have $\lfloor \frac{400}{32} \rfloor = 12$, $\lfloor \frac{400}{81} \rfloor = 4$, $\lfloor \frac{400}{343} \rfloor = 1$, and $\lfloor \frac{400}{400} \rfloor = 1$ multiples, respectively, in the range $1 \le N \le 400$. Because there are no common multiples, the required answer is $12 + 4 + 1 + 1 = 18$.

2002B 25. **(E)** Note that

$$f(x) + f(y) = x^2 + 6x + y^2 + 6y + 2 = (x+3)^2 + (y+3)^2 - 16$$

and

$$f(x) - f(y) = x^2 - y^2 + 6(x - y) = (x - y)(x + y + 6).$$

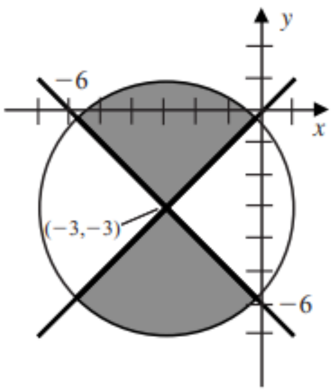
The given conditions can be written as

$$(x+3)^2 + (y+3)^2 \le 16 \quad \text{and} \quad (x-y)(x+y+6) \le 0.$$

The first inequality describes the region on and inside the circle of radius 4 with center $(-3, -3)$. The second inequality can be rewritten as

$$(x - y \ge 0 \text{ and } x + y + 6 \le 0) \quad \text{or} \quad (x - y \le 0 \text{ and } x + y + 6 \ge 0).$$

Each of these inequalities describes a half-plane bounded by a line that passes through $(-3, -3)$ and has slope 1 or -1. Thus, the set R is the shaded region in the following diagram, and its area is half the area of the circle, which is $8\pi \approx 25.13$.



2003A

25. (C) The domain of f is $\{x \mid ax^2 + bx \geq 0\}$. If $a = 0$, then for every positive value of b, the domain and range of f are each equal to the interval $[0, \infty)$, so 0 is a possible value of a.
If $a \neq 0$, the graph of $y = ax^2 + bx$ is a parabola with x-intercepts at $x = 0$ and $x = -b/a$. If $a > 0$, the domain of f is $(-\infty, -b/a] \cup [0, \infty)$, but the range of f cannot contain negative numbers. If $a < 0$, the domain of f is $[0, -b/a]$. The maximum value of f occurs halfway between the x-intercepts, at $x = -b/2a$, and

$$f\left(-\frac{b}{2a}\right) = \sqrt{a\left(\frac{b^2}{4a^2}\right) + b\left(-\frac{b}{2a}\right)} = \frac{b}{2\sqrt{-a}}.$$

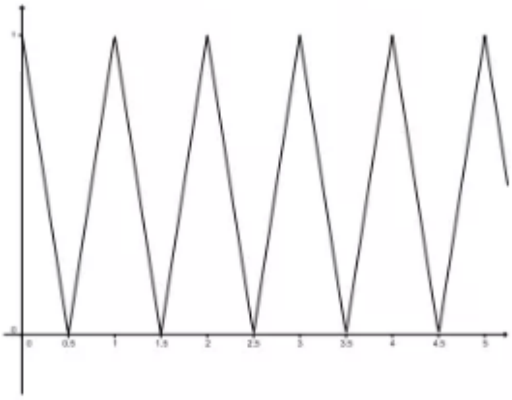
Hence, the range of f is $[0, b/2\sqrt{-a}]$. For the domain and range to be equal, we must have

$$-\frac{b}{a} = \frac{b}{2\sqrt{-a}} \quad \text{so} \quad 2\sqrt{-a} = -a.$$

The only solution is $a = -4$. Thus there are two possible values of a, and they are $a = 0$ and $a = -4$.

2012A

25. **Answer (C):** Because $-1 \le 2\{x\} - 1 \le 1$ it follows that $0 \le f(x) \le 1$ for all $x \in \mathbb{R}$. Thus $0 \le nf(xf(x)) \le n$, and therefore all real solutions x of the required equation are in the interval $[0, n]$. Also $f(x)$ is periodic with period 1, $f(x) = 1 - 2x$ if $0 \le x \le \frac{1}{2}$, and $f(x) = 2x - 1$ if $\frac{1}{2} \le x \le 1$. Thus the graph of $y = f(x)$ for $x \ge 0$ consists of line segments joining the points with coordinates $(k, 1)$, $(k + \frac{1}{2}, 0)$, $(k + 1, 1)$ for integers $k \ge 0$ as shown.



Let a be an integer such that $0 \le a \le n - 1$. Consider the interval $[a, a + \frac{1}{2})$. If $x \in [a, a + \frac{1}{2})$, then $f(x) = |2\{x\} - 1| = |2(x - a) - 1| = 1 + 2a - 2x$ and thus $g(x) := xf(x) = x(1 + 2a - 2x)$. Suppose $a \ge 1$ and $a \le x < y < a + \frac{1}{2}$. Then $2x + 2y - 2a - 1 > 2a - 1 \ge 1$ and so $(y - x)(2x + 2y - 2a - 1) > 0$, which is equivalent to $g(x) = x(1 + 2a - 2x) > y(1 + 2a - 2y) = g(y)$. Thus g is strictly decreasing on $[a, a + \frac{1}{2})$ and so it maps $[a, a + \frac{1}{2})$ bijectively to $(0, a]$. Thus the graph of the function $y = f(g(x))$ on the interval $[a, a + \frac{1}{2})$ oscillates from 1 to 0 as many times as the graph of the function $y = f(x)$ on the interval $(0, a]$. It follows that the line with equation $y = \frac{x}{n}$ intersects the graph of $y = f(g(x))$ on the interval $[a, a + \frac{1}{2})$ exactly $2a$ times.

If $a = 0$ and $x \in [a, a + \frac{1}{2})$, then $g(x) = x(1 - 2x)$ satisfies $0 \le g(x) \le \frac{1}{8}$, so $f(g(x)) = 1 - 2g(x) = 4x^2 - 2x + 1$. If $x \in [0, \frac{1}{2})$ and $n \ge 1$, then $0 \le \frac{x}{n} < \frac{1}{2n} \le \frac{1}{2}$. Because $\frac{1}{2} \le 1 - 2g(x) \le 1$, it follows that the parabola $y = f(g(x))$ does not intersect any of the lines with equation $y = \frac{x}{n}$ on the interval $[0, \frac{1}{2})$.

Similarly, if $x \in [a + \frac{1}{2}, a + 1)$, then $f(x) = |2\{x\} - 1| = |2(x - a) - 1| = 2x - 2a - 1$ and $g(x) := xf(x) = x(2x - 2a - 1)$. This time if $a + \frac{1}{2} \le x < y < a + 1$, then $2x + 2y - 2a + 1 \ge 2a + 1 \ge 1$ and so $(x - y)(2x + 2y - 2a + 1) < 0$, which is equivalent to $g(x) < g(y)$. Thus g is strictly increasing on $[a + \frac{1}{2}, a + 1)$ and so it maps $[a + \frac{1}{2}, a + 1)$ bijectively to $[0, a + 1)$. Thus the graph of the function $y = f(g(x))$ on the interval $[a + \frac{1}{2}, a + 1)$ oscillates as many times as the graph of $y = f(x)$ on the interval $[0, a + 1)$. It follows that the line with equation $y = \frac{x}{n}$ intersects the graph of $y = f(g(x))$ on the interval $[a + \frac{1}{2}, a + 1)$ exactly $2(a + 1)$ times. Therefore the total number of intersections of the line $y = \frac{x}{n}$ and the graph of $y = f(g(x))$ is equal to

$$\sum_{a=0}^{n-1}(2a + 2(a + 1)) = 2\sum_{a=0}^{n-1}(2a + 1) = 2n^2.$$

Finally the smallest n such that $2n^2 \ge 2012$ is $n = 32$ because $2 \cdot 31^2 = 1922$ and $2 \cdot 32^2 = 2048$.