UNIT 15 EXERCISES 21-25

POLYNOMIALS

2003A

21. (D) Since $P(0) = 0$, we have $e = 0$ and $P(x) = x(x^4 + ax^3 + bx^2 + cx + d)$. Suppose that the four remaining x-intercepts are at p, q, r, and s. Then

$$x^4 + ax^3 + bx^2 + cx + d = (x - p)(x - q)(x - r)(x - s),$$

and $d = pqrs \neq 0$.
Any of the other constants could be zero. For example, consider

$$P_1(x) = x^5 - 5x^3 + 4x = x(x + 2)(x + 1)(x - 1)(x - 2)$$

and

$$P_2(x) = x^5 - 5x^4 + 20x^2 - 16x = x(x + 2)(x - 1)(x - 2)(x - 4).$$

OR

Since $P(0) = 0$, we must have $e = 0$, so

$$P(x) = x(x^4 + ax^3 + bx^2 + cx + d).$$

If $d = 0$, then

$$P(x) = x(x^4 + ax^3 + bx^2 + cx) = x^2(x^3 + ax^2 + bx + c),$$

which has a double root at $x = 0$. Hence $d \neq 0$.

OR

There is also a calculus-based solution. Since $P(x)$ has five distinct zeros and $x = 0$ is one of the zeros, it must be a zero of multiplicity one. This is equivalent to having $P(0) = 0$, but $P'(0) \neq 0$. Since

$$P'(x) = 5x^4 + 4ax^3 + 3bx^2 + 2cx + d, \quad \text{we must have} \quad 0 \neq P'(0) = d.$$

2004B

21. **(C)** A line $y = mx$ intersects the ellipse in 0, 1, or 2 points. The intersection consists of exactly one point if and only if $m = a$ or $m = b$. Thus a and b are the values of m for which the system

$$\begin{aligned} 2x^2 + xy + 3y^2 - 11x - 20y + 40 &= 0 \\ y &= mx \end{aligned}$$

has exactly one solution. Substituting mx for y in the first equation gives

$$2x^2 + mx^2 + 3m^2x^2 - 11x - 20mx + 40 = 0,$$

or, by rearranging the terms,

$$(3m^2 + m + 2)x^2 - (20m + 11)x + 40 = 0.$$

The discriminant of this equation is

$$(20m + 11)^2 - 4 \cdot 40 \cdot (3m^2 + m + 2) = -80m^2 + 280m - 199,$$

which must be zero if $m = a$ or $m = b$. Thus $a + b$ is the sum of the roots of the equation $-80m^2 + 280m - 199 = 0$, which is $\frac{280}{80} = \frac{7}{2}$.

2007A

21. **Answer (A):** The product of the zeros of f is c/a, and the sum of the zeros is $-b/a$. Because these two numbers are equal, $c = -b$, and the sum of the coefficients is $a + b + c = a$, which is the coefficient of x^2. To see that none of the other choices is correct, let $f(x) = -2x^2 - 4x + 4$. The zeros of f are $-1 \pm \sqrt{3}$, so the sum of the zeros, the product of the zeros, and the sum of the coefficients are all -2. However, the coefficient of x is -4, the y-intercept is 4, the x-intercepts are $-1 \pm \sqrt{3}$, and the mean of the x-intercepts is -1.

2010A 21. **Answer (A):** Let the three points of intersections have x-coordinates p, q, and r, and let $f(x) = x^6 - 10x^5 + 29x^4 - 4x^3 + ax^2 - bx - c$. Then $f(p) = f(q) = f(r) = 0$, and $f(x) \geq 0$ for all x, so $f(x) = ((x-p)(x-q)(x-r))^2 = (x^3 - Ax^2 + Bx - C)^2$, where $A = p+q+r$, $B = pq+qr+rp$, and $C = pqr$. The coefficient of x^5 is $-10 = -2A$, so $A = 5$. The coefficient of x^4 is $29 = A^2+2B = 25 + 2B$, so $B = 2$. The coefficient of x^3 is $-4 = -2C - 2AB = -2C - 20$, so $C = -8$. Thus $f(x) = (x^3 - 5x^2 + 2x + 8)^2$. Because the sums of the coefficients of the even and odd powers of x are equal, -1 is a zero of $f(x)$. Factoring gives $f(x) = ((x+1)(x^2 - 6x + 8))^2 = ((x+1)(x-2)(x-4))^2$, and the largest of the three zeros is 4.

2010B 21. **Answer (B):** Because $1, 3, 5$, and 7 are roots of the polynomial $P(x) - a$, it follows that

$$P(x) - a = (x-1)(x-3)(x-5)(x-7)Q(x),$$

where $Q(x)$ is a polynomial with integer coefficients. The previous identity must hold for $x = 2, 4, 6$, and 8, thus

$$-2a = -15Q(2) = 9Q(4) = -15Q(6) = 105Q(8).$$

Therefore $315 = \text{lcm}(15, 9, 105)$ divides a, that is a is an integer multiple of 315. Let $a = 315A$. Because $Q(2) = Q(6) = 42A$, it follows that $Q(x) - 42A = (x-2)(x-6)R(x)$ where $R(x)$ is a polynomial with integer coefficients. Because $Q(4) = -70A$ and $Q(8) = -6A$ it follows that $-112A = -4R(4)$ and $-48A = 12R(8)$, that is $R(4) = 28A$ and $R(8) = -4A$. Thus $R(x) = 28A + (x-4)(-6A + (x-8)T(x))$ where $T(x)$ is a polynomial with integer coefficients. Moreover, for any polynomial $T(x)$ and any integer A, the polynomial $P(x)$ constructed this way satisfies the required conditions. The required minimum is obtained when $A = 1$ and so $a = 315$.

2014A

21. **Answer (A):** If $x = n + r$, where n is an integer, $1 \le n \le 2013$, and $0 \le r < 1$, then $f(x) = n(2014^r - 1)$. The condition $f(x) \le 1$ is equivalent to $2014^r \le 1 + \frac{1}{n}$, or $0 \le r \le \log_{2014}\left(\frac{n+1}{n}\right)$. Thus the required sum is

$$\log_{2014}\frac{2}{1} + \log_{2014}\frac{3}{2} + \log_{2014}\frac{4}{3} + \cdots + \log_{2014}\frac{2014}{2013}$$
$$= \log_{2014}\left(\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{2014}{2013}\right) = \log_{2014}(2014) = 1.$$

2018A

21. **Answer (B):** By Descartes' Rule of Signs, none of these polynomials has a positive root, and each one has exactly one negative root. Because each polynomial is positive at $x = 0$ and negative at $x = -1$, it follows that each has exactly one root between -1 and 0. Note also that each polynomial is increasing throughout the interval $(-1, 0)$. Because $x^{19} > x^{17}$ for all x in the interval $(-1, 0)$, it follows that the polynomial in choice **A** is greater than the polynomial in choice **B** on that interval, which implies that the root of the polynomial in choice **A** is less than the root of the polynomial in choice **B**. Because $x^{13} > x^{11}$ for all x in the interval $(-1, 0)$, it follows that the polynomial in choice **C** is greater than the polynomial in choice **A** on that interval, which implies that the root of the polynomial in choice **C** is less than the root of the polynomial in choice **A** and therefore less than the root of the polynomial in choice **B**. The same reasoning shows that the root of the polynomial in choice **D** is less than the root of the polynomial in choice **B**.

Furthermore, $2018 > 2018x^6$ on the interval $(-1, 0)$, so $x^6 + 2018 > 2019x^6$, from which it follows that $x^{11}(x^6 + 2018) < 2019x^{17}$. Therefore the polynomial in choice **B** is less than $2019x^{17} + 1$ on the interval $(-1, 0)$. The polynomial in choice **E** has root $-\left(1 - \frac{1}{2019}\right)$. Bernoulli's Inequality shows that $(1 + x)^{17} > 1 + 17x$ for all $x > -1$, which implies that

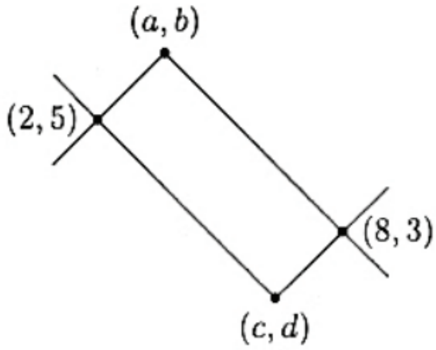
$$-2019\left(1 - \frac{1}{2019}\right)^{17} + 1 < -2019\left(1 - \frac{17}{2019}\right) + 1 = -2001 < 0,$$

so the polynomial in choice **B** is negative at the root of the polynomial in choice **E**. This shows that the root of the polynomial in choice **B** is greater than the root in choice **E**.

Because the unique real root of the polynomial in choice **B** is greater than the unique root of the polynomial in each of the other choices, that polynomial has the greatest real root.

1999

22. **(C)** The first graph is an inverted 'V-shaped' right angle with vertex at (a, b) and the second is a V-shaped right angle with vertex at (c, d). Thus $(a, b), (2, 5), (c, d)$, and $(8, 3)$ are consecutive vertices of a rectangle. The diagonals of this rectangle meet at their common midpoint, so the x-coordinate of this midpoint is $(2+8)/2 = (a+c)/2$. Thus $a+c = 10$.



OR

Use the given information to obtain the equations $5 = -|2-a| + b$, $5 = |2-c| + d$, $3 = -|8-a| + b$, and $3 = |8-c| + d$. Subtract the third from the first to eliminate b and subtract the fourth from the second to eliminate d. The two resulting equations $|8-a| - |2-a| = 2$ and $|2-c| - |8-c| = 2$ can be solved for a and c. To solve the former, first consider all $a \leq 2$, for which the equation reduces to $8 - a - (2-a) = 2$, which has no solutions. Then consider all a in the interval $2 \leq a \leq 8$, for which the equation reduces to $8 - a - (a-2) = 2$, which yields $a = 4$. Finally, consider all $a \geq 8$, for which the equation reduces to $a - 8 - (a-2) = 2$, which has no solutions. The other equation can be solved similarly to show that $c = 6$. Thus $a + c = 10$.

2010A

22. **Answer (A):** Note that

$$f(x) = \begin{cases} -(x-1)-(2x-1)-\cdots-(119x-1), & \text{if } x \le \frac{1}{119}; \\ -(x-1)-(2x-1)-\cdots-((m-1)x-1) & \\ \quad +(mx-1)+\cdots+(119x-1), & \text{if } \frac{1}{m} \le x \le \frac{1}{m-1};\ 2 \le m \le \\ (x-1)+(2x-1)+\cdots+(119x-1), & \text{if } x \ge 1. \end{cases}$$

The graph of $f(x)$ consists of a negatively sloped ray for $x \le \frac{1}{119}$, a positively sloped ray for $x \ge 1$, and for $\frac{1}{119} \le x \le 1$ a sequence of line segments whose slopes increase as x increases. The minimum value of $f(x)$ occurs at the right endpoint of the rightmost interval in which the graph has a non-positive slope. The slope on the interval $[\frac{1}{m}, \frac{1}{m-1}]$ is

$$\sum_{k=m}^{119} k - \sum_{k=1}^{m-1} k = \sum_{k=1}^{119} k - 2\sum_{k=1}^{m-1} k = 7140 - (m-1)(m).$$

The inequality $7140 + m - m^2 \le 0$ is satisfied in the interval $[-84, 85]$ with equality at the endpoints. Therefore on the interval $\left[\frac{1}{85}, \frac{1}{84}\right]$ the graph of $f(x)$ has a slope of 0 and a constant value of $(84)(1) + (119-84)(-1) = 49$.

2018B 22. **Answer (D):** Let $P(x) = ax^3 + bx^2 + cx + d$, where a, b, c, and d are integers between 0 and 9, inclusive. The condition $P(-1) = -9$ is equivalent to $-a + b - c + d = -9$. Adding 18 to both sides gives $(9 - a) + b + (9 - c) + d = 9$ where $0 \le 9 - a, b, 9 - c, d \le 9$. By the stars and bars argument, there are $\binom{9+4-1}{4-1} = \binom{12}{3} = 220$ nonnegative integer solutions to $x_1 + x_2 + x_3 + x_4 = 9$. Each of these give rise to one of the desired polynomials.

OR

With the notation above, note that $(a + c) - (b + d) = 9$ can occur in several ways: $b+d = k$, $a+c = 9+k$ where $k = 0, 1, 2, \ldots, 9$. There are $k+1$ solutions to $b+d = k$ and $10-k$ solutions to $a+c = 9+k$ under the restrictions on a, b, c, and d, yielding $\sum_{k=0}^{9}(k + 1)(10 - k) = 220$ solutions in all.

2004B 23. **(C)** Let a denote the zero that is an integer. Because the coefficient of x^3 is 1, there can be no other rational zeros, so the two other zeros must be $\frac{a}{2} \pm r$ for some irrational number r. The polynomial is then

$$(x-a)\left[x-\left(\frac{a}{2}+r\right)\right]\left[x-\left(\frac{a}{2}-r\right)\right]$$
$$= x^3 - 2ax^2 + \left(\frac{5}{4}a^2 - r^2\right)x - a\left(\frac{1}{4}a^2 - r^2\right).$$

Therefore $a = 1002$ and the polynomial is

$$x^3 - 2004x^2 + (5(501)^2 - r^2)x - 1002((501)^2 - r^2).$$

All coefficients are integers if and only if r^2 is an integer, and the zeros are positive and distinct if and only if $1 \leq r^2 \leq 501^2 - 1 = 251{,}000$. Because r cannot be an integer, there are $251{,}000 - 500 = 250{,}500$ possible values of n.

2010B

23. **Answer (A):** Because both $P(Q(x))$ and $Q(P(x))$ have four distinct real zeros, both $P(x)$ and $Q(x)$ must have two distinct real zeros, so there are real numbers h_1, k_1, h_2, and k_2 such that $P(x) = (x-h_1)^2-k_1^2$ and $Q(x) = (x-h_2)^2-k_2^2$. The zeros of $P(Q(x))$ occur when $Q(x) = h_1 \pm k_1$. The solutions of each equation are equidistant from h_2, so $h_2 = -19$. It follows that $Q(-15) - Q(-17) = (16-k_2^2)-(4-k_2^2) = 12$, and also $Q(-15)-Q(-17) = 2k_1$, so $k_1 = 6$. Similarly $h_1 = -54$, so $2k_2 = P(-49) - P(-51) = (25-k_1^2) - (9-k_1^2) = 16$, and $k_2 = 8$. Thus the sum of the minimum values of $P(x)$ and $Q(x)$ is $-k_1^2 - k_2^2 = -100$.

2017A

23. **Answer (C):** Let q be the additional root of $f(x)$. Then

$$\begin{aligned} f(x) &= (x-q)(x^3+ax^2+x+10) \\ &= x^4+(a-q)x^3+(1-qa)x^2+(10-q)x-10q. \end{aligned}$$

Thus $100 = 10-q$, so $q = -90$ and $c = -10q = 900$. Also $1 = a-q = a+90$, so $a = -89$. It follows, using the factored form of f shown above, that $f(1) = \big(1-(-90)\big)\cdot(1-89+1+10) = 91\cdot(-77) = -7007$.

2017B

23. **Answer (D):** Let $g(x) = f(x) - x^2$. Then $g(2) = g(3) = g(4) = 0$, so for some constant $a \neq 0$, $g(x) = a(x-2)(x-3)(x-4)$. Thus the coefficients of x^3 and x^2 in $f(x)$ are a and $1 - 9a$, respectively, so the sum of the roots of $f(x)$ is $9 - \frac{1}{a}$. If $L(x)$ is any linear function, then the roots of $f(x) - L(x)$ have the same sum. The given information implies that the sets of roots for three such functions are $\{2, 3, x_1\}$, $\{2, 4, x_2\}$, and $\{3, 4, x_3\}$, where

$$24 = x_1 + x_2 + x_3 = 3\left(9 - \frac{1}{a}\right) - 2(2+3+4) = 9 - \frac{3}{a},$$

so $a = -\frac{1}{5}$. Therefore $f(x) = x^2 - \frac{1}{5}(x-2)(x-3)(x-4)$, and $f(0) = \frac{24}{5}$. (In fact, $D = (9, 39)$, $E = (8, 40)$, $F = (7, 37)$, and the roots of f are 12, $1+i$, and $1-i$.)

2005A

24. **(B)** The polynomial $P(x) \cdot R(x)$ has degree 6, so $Q(x)$ must have degree 2. Therefore Q is uniquely determined by the ordered triple $(Q(1), Q(2), Q(3))$. When $x = 1$, 2, or 3, we have $0 = P(x) \cdot R(x) = P(Q(x))$. It follows that $(Q(1), Q(2), Q(3))$ is one of the 27 ordered triples (i, j, k), where i, j, and k can be chosen from the set $\{1, 2, 3\}$. However, the choices $(1,1,1)$, $(2,2,2)$, $(3,3,3)$, $(1,2,3)$, and $(3,2,1)$ lead to polynomials $Q(x)$ defined by $Q(x) = 1$, 2, 3, x, and $4-x$, respectively, all of which have degree less than 2. The other 22 choices for $(Q(1), Q(2), Q(3))$ yield non-collinear points, so in each case $Q(x)$ is a quadratic polynomial.

2016A

24. **Answer (B):** Because a and b are positive, all the roots must be positive. Let the roots be r, s, and t. Then

$$x^3 - ax^2 + bx - a = (x-r)(x-s)(x-t) = x^3 - (r+s+t)x^2 + (rs+st+tr)x - rst.$$

Therefore $r + s + t = a = rst$. The Arithmetic Mean–Geometric Mean Inequality implies that $27rst \le (r+s+t)^3 = (rst)^3$, from which $a = rst \ge 3\sqrt{3}$. Furthermore, equality is achieved if and only if $r = s = t = \sqrt{3}$. In this case $b = rs + st + tr = 9$.

2002A

25. **(B)** The sum of the coefficients of P and the sum of the coefficients of Q will be equal, so $P(1) = Q(1)$. The only answer choice with an intersection at $x = 1$ is (B). (The polynomials in graph B are $P(x) = 2x^4 - 3x^2 - 3x - 4$ and $Q(x) = -2x^4 - 2x^2 - 2x - 2$.)

2014A 25. **Answer (B):** Let $O = (0, 0)$, $A = (4, 3)$, and $B = (-4, -3)$. Because $A, B \in P$ and O is the midpoint of $\overline{AB}$, it follows that $\overline{AB}$ is the latus rectum of the parabola P. Thus the directrix is parallel to $\overline{AB}$. Let T be the foot of the perpendicular from O to the directrix of P. Because $OT = OA = OB = 5$ and $\overline{OT}$ is perpendicular to $\overline{AB}$, it follows that $T = (3, -4)$. Thus the equation of the directrix is $y+4 = \frac{3}{4}(x-3)$, and in general form the equation is $4y-3x+25 = 0$.

Using the formula for the distance from a point to a line, as well as the definition of P as the locus of points equidistant from O and the directrix, the equation of P is

$$\sqrt{x^2 + y^2} = \frac{|4y - 3x + 25|}{\sqrt{4^2 + 3^2}}.$$

After squaring and rearranging, this is equivalent to

$$\begin{aligned} 25x^2 + 25y^2 &= 25(x^2 + y^2) = (4y - 3x + 25)^2 \\ &= 16y^2 + 9x^2 - 24xy + 25^2 + 50(4y - 3x), \end{aligned}$$

and

$$(4x + 3y)^2 = 25(25 + 2(4y - 3x)). \qquad (1)$$

Assume x and y are integers. Then $4x + 3y$ is divisible by 5. If $4x + 3y = 5s$ for $s \in \mathbb{Z}$, then $2s^2 = 50+16y-12x = 50+16y-3(5s-3y) = 50+25y-15s$. Thus s is divisible by 5. If $s = 5t$ for $t \in \mathbb{Z}$, then $2t^2 = 2+y-3t$, and so $y = 2t^2+3t-2$. In addition $4x = 5s - 3y = 25t - 3y = 25t - 3(2t^2 + 3t - 2) = -6t^2 + 16t + 6$, and thus t is odd. If $t = 2u + 1$ for $u \in \mathbb{Z}$, then

$$x = -6u^2 + 2u + 4 \text{ and } y = 8u^2 + 14u + 3. \qquad (2)$$

Conversely, if x and y are defined as in (2) for $u \in \mathbb{Z}$, then x and y are integers and they satisfy (1), which is the equation of P. Lastly, with $u \in \mathbb{Z}$,

$$\begin{aligned} |4x + 3y| &= \left|-24u^2 + 8u + 16 + 24u^2 + 42u + 9\right| \\ &= |50u + 25| \le 1000 \end{aligned}$$

if and only if u is an integer such that $|2u + 1| \le 39$. That is, $-20 \le u \le 19$, and so the required answer is $19 - (-21) = 40$.