UNIT 14 EXERCISES 21-25

SEQUENCE

2002A 21. **(B)** Writing out more terms of the sequence yields

$$4,\ 7,\ 1,\ 8,\ 9,\ 7,\ 6,\ 3,\ 9,\ 2,\ 1,\ 3,\ 4,\ 7,\ 1\ldots.$$

The sequence repeats itself, starting with the 13th term. Since $S_{12} = 60$, $S_{12k} = 60k$ for all positive integers k. The largest k for which $S_{12k} \le 10,000$ is

$$k = \left\lfloor \frac{10,000}{60} \right\rfloor = 166,$$

and $S_{12\cdot 166} = 60 \cdot 166 = 9960$. To have $S_n > 10,000$, we need to add enough additional terms for their sum to exceed 40. This can be done by adding the next 7 terms of the sequence, since their sum is 42. Thus, the smallest value of n is $12 \cdot 166 + 7 = 1999$.

2002B 21. **(A)** Since $2002 = 11 \cdot 13 \cdot 14$, we have

$$a_n = \begin{cases} 11, & \text{if } n = 13 \cdot 14 \cdot i, \text{ where } i = 1, 2, \ldots, 10; \\ 13, & \text{if } n = 14 \cdot 11 \cdot j, \text{ where } j = 1, 2, \ldots, 12; \\ 14, & \text{if } n = 11 \cdot 13 \cdot k, \text{ where } k = 1, 2, \ldots, 13; \\ 0, & \text{otherwise.} \end{cases}$$

Hence $\displaystyle\sum_{n=1}^{2001} a_n = 11 \cdot 10 + 13 \cdot 12 + 14 \cdot 13 = 448.$

2006A

23. **(B)** For every sequence $S = (a_1, a_2, \ldots, a_n)$ of at least three terms,

$$A^2(S) = \left(\frac{a_1 + 2a_2 + a_3}{4}, \frac{a_2 + 2a_3 + a_4}{4}, \ldots, \frac{a_{n-2} + 2a_{n-1} + a_n}{4}\right).$$

Thus for $m = 1$ and 2, the coefficients of the terms in the numerator of $A^m(S)$ are the binomial coefficients $\binom{m}{0}, \binom{m}{1}, \ldots, \binom{m}{m}$, and the denominator is 2^m. Because $\binom{m}{r} + \binom{m}{r+1} = \binom{m+1}{r+1}$ for all integers $r \geq 0$, the coefficients of the terms in the numerators of $A^{m+1}(S)$ are $\binom{m+1}{0}, \binom{m+1}{1}, \ldots, \binom{m+1}{m+1}$ for $2 \leq m \leq n - 2$. The definition implies that the denominator of each term in $A^{m+1}(S)$ is 2^{m+1}. For the given sequence, the sole term in $A^{100}(S)$ is

$$\frac{1}{2^{100}} \sum_{m=0}^{100} \binom{100}{m} a_{m+1} = \frac{1}{2^{100}} \sum_{m=0}^{100} \binom{100}{m} x^m = \frac{1}{2^{100}} (x + 1)^{100}.$$

Therefore

$$\left(\frac{1}{2^{50}}\right) = A^{100}(S) = \left(\frac{(1 + x)^{100}}{2^{100}}\right),$$

so $(1 + x)^{100} = 2^{50}$, and because $x > 0$, we have $x = \sqrt{2} - 1$.

2001

25. **(D)** If a, b, and c are three consecutive terms of such a sequence, then $ac-1=b$, which can be rewritten as $c=(1+b)/a$. Applying this rule recursively and simplifying yields

$$\ldots, a, b, \frac{1+b}{a}, \frac{1+a+b}{ab}, \frac{1+a}{b}, a, b, \ldots$$

This shows that at most five different terms can appear in such a sequence. Moreover, the value of a is determined once the value 2000 is assigned to b and the value 2001 is assigned to another of the first five terms. Thus, there are four such sequences that contain 2001 s a term, namely

$$2001, 2000, 1, \frac{1}{1000}, \frac{1001}{1000}, 2001, \ldots,$$
$$1, 2000, 2001, \frac{1001}{1000}, \frac{1}{1000}, 1, \ldots,$$
$$\frac{2001}{4001999}, 2000, 4001999, 2001, \frac{2002}{4001999}, \frac{2001}{4001999}, \ldots, \text{ and}$$
$$4001999, 2000, \frac{2001}{4001999}, \frac{2002}{4001999}, 2001, 4001999, \ldots,$$

respectively. The four values of x are 2001, 1, $\frac{2001}{4001999}$, and 4001999.

2006B 24. **(C)** For a fixed value of y, the values of $\sin x$ for which $\sin^2 x - \sin x \sin y + \sin^2 y = \frac{3}{4}$ can be determined by the quadratic formula. Namely,

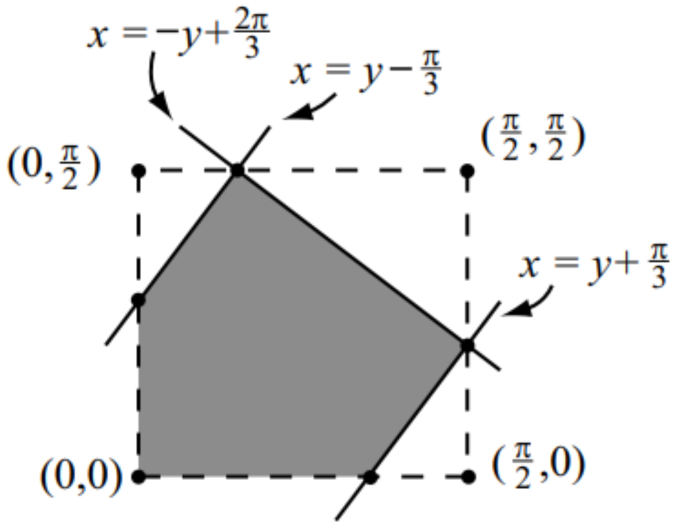
$$\sin x = \frac{\sin y \pm \sqrt{\sin^2 y - 4(\sin^2 y - \frac{3}{4})}}{2} = \frac{1}{2}\sin y \pm \frac{\sqrt{3}}{2}\cos y.$$

Because $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, this implies that

$$\sin x = \cos\left(\frac{\pi}{3}\right)\sin y \pm \sin\left(\frac{\pi}{3}\right)\cos y = \sin\left(y \pm \frac{\pi}{3}\right).$$

Within S, $\sin x = \sin(y - \frac{\pi}{3})$ implies $x = y - \frac{\pi}{3}$. However, the case $\sin x = \sin(y + \frac{\pi}{3})$ implies $x = y + \frac{\pi}{3}$ when $y \le \frac{\pi}{6}$, and $x = -y + \frac{2\pi}{3}$ when $y \ge \frac{\pi}{6}$. Those three lines divide the region S into four subregions, within each of which the truth value of the inequality is constant. Testing the points $(0,0)$, $(\frac{\pi}{2}, 0)$, $(0, \frac{\pi}{2})$, and $(\frac{\pi}{2}, \frac{\pi}{2})$ shows that the inequality is true only in the shaded subregion. The area of this subregion is

$$\left(\frac{\pi}{2}\right)^2 - \frac{1}{2}\cdot\left(\frac{\pi}{3}\right)^2 - 2\cdot\frac{1}{2}\cdot\left(\frac{\pi}{6}\right)^2 = \frac{\pi^2}{6}.$$



2008A 25. **Answer (D):** Let $z_n = a_n + b_n i$. Then

$$\begin{aligned} z_{n+1} &= (\sqrt{3}a_n - b_n) + (\sqrt{3}b_n + a_n)i = (a_n + b_n i)(\sqrt{3} + i) \\ &= z_n(\sqrt{3} + i) = z_1(\sqrt{3} + i)^n. \end{aligned}$$

Noting that $\sqrt{3} + i = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$ and applying DeMoivre's formula gives

$$\begin{aligned} 2 + 4i = z_{100} &= z_1\left(2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)\right)^{99} \\ &= z_1 \cdot 2^{99}\left(\cos\left(\frac{99\pi}{6}\right) + i\sin\left(\frac{99\pi}{6}\right)\right) \\ &= (a_1 + b_1 i) \cdot 2^{99} \cdot i = -2^{99}b_1 + 2^{99}a_1 i. \end{aligned}$$

So $2 = -2^{99}b_1$, $4 = 2^{99}a_1$, and

$$a_1 + b_1 = \frac{4}{2^{99}} - \frac{2}{2^{99}} = \frac{1}{2^{98}}.$$

OR

Note that

$$\begin{aligned} (a_{n+2}, b_{n+2}) &= \Big(\sqrt{3}(\sqrt{3}a_n - b_n) - (\sqrt{3}b_n + a_n), \\ &\qquad \sqrt{3}(\sqrt{3}b_n + a_n) + (\sqrt{3}a_n - b_n)\Big) \\ &= \left(-2\sqrt{3}b_n + 2a_n, 2\sqrt{3}a_n + 2b_n\right), \\ (a_{n+3}, b_{n+3}) &= \Big(\sqrt{3}(-2\sqrt{3}b_n + 2a_n) - (2\sqrt{3}a_n + 2b_n), \\ &\qquad \sqrt{3}(2\sqrt{3}a_n + 2b_n) + (-2\sqrt{3}b_n + 2a_n)\Big) \\ &= 8(-b_n, a_n), \end{aligned}$$

and $(a_{n+6}, b_{n+6}) = 8(-b_{n+3}, a_{n+3}) = -64(a_n, b_n)$. Because $97 = 1 + 16 \cdot 6$, we have

$$(a_{97}, b_{97}) = (-64)^{16}(a_1, b_1) = 2^{96}(a_1, b_1)$$

and

$$(2, 4) = (a_{100}, b_{100}) = 2^3\,(-b_{97}, a_{97}) = 2^{99}(-b_1, a_1).$$

The conclusion follows as in the first solution.

2009A 25. **Answer (A):** Recognize the similarity between the recursion formula given and the trigonometric identity

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}.$$

Also note that the first two terms of the sequence are tangents of familiar angles, namely $\frac{\pi}{4}$ and $\frac{\pi}{6}$. Let $c_1 = 3$, $c_2 = 2$, and $c_{n+2} = (c_n + c_{n+1}) \bmod 12$. We claim that the sequence $\{a_n\}$ satisfies $a_n = \tan\left(\frac{\pi c_n}{12}\right)$. Note that

$$a_1 = 1 = \tan\left(\frac{\pi}{4}\right) = \tan\left(\frac{\pi c_1}{12}\right) \text{ and}$$
$$a_2 = \frac{1}{\sqrt{3}} = \tan\left(\frac{\pi}{6}\right) = \tan\left(\frac{\pi c_2}{12}\right).$$

By induction on n, the formula for the tangent of the sum of two angles, and the fact that the period of $\tan x$ is π,

$$a_{n+2} = \frac{a_n + a_{n+1}}{1 - a_n a_{n+1}} = \frac{\tan\left(\frac{\pi c_n}{12}\right) + \tan\left(\frac{\pi c_{n+1}}{12}\right)}{1 - \tan\left(\frac{\pi c_n}{12}\right)\tan\left(\frac{\pi c_{n+1}}{12}\right)}$$
$$= \tan\left(\frac{\pi(c_n + c_{n+1})}{12}\right) = \tan\left(\frac{\pi c_{n+2}}{12}\right).$$

The first few terms of the sequence $\{c_n\}$ are:

$$3, 2, 5, 7, 0, 7, 7, 2, 9, 11, 8, 7, 3, 10, 1, 11, 0, 11, 11, 10, 9, 7, 4, 11, 3, 2.$$

So the sequence c_n is periodic with period 24. Because $2009 = 24 \cdot 83 + 17$, it follows that $c_{2009} = c_{17} = 0$. Thus $|a_{2009}| = |\tan\left(\frac{\pi c_{17}}{12}\right)| = 0$.

2016B 25. **Answer (A):** Express each term of the sequence (a_n) as $2^{\frac{b_n}{19}}$. (Equivalently, let b_n be the logarithm of a_n to the base $\sqrt[19]{2}$.) The recursive definition of the sequence (a_n) translates into $b_0 = 0$, $b_1 = 1$, and $b_n = b_{n-1} + 2b_{n-2}$ for $n \geq 2$. Then the product $a_1a_2 \cdots a_k$ is an integer if and only if $\sum_{i=1}^{k} b_i$ is divisible by 19. Let $c_n = b_n \bmod 19$. It follows that $a_1a_2 \cdots a_k$ is an integer if and only if $p_k = \sum_{i=1}^{k} c_i$ is divisible by 19. Let $q_k = p_k \bmod 19$. Because the largest answer choice is 21, it suffices to compute c_k and q_k successively for k from 1 up to at most 21, until q_k first equals 0. The modular computations are straightforward from the definitions.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
c_k	1	1	3	5	11	2	5	9	0	18	18	16	14	8	17	14	10
q_k	1	2	5	10	2	4	9	18	18	17	16	13	8	16	14	9	0

Thus the requested answer is 17.

OR

Using standard techniques, the recurrence relation for b_n can be solved to get $b_n = \frac{1}{3}(2^n - (-1)^n)$. Let $S_k = b_1 + b_2 + \cdots + b_k$. Then it is straightforward to show that $S_k = \frac{1}{3}(2^{k+1} - 1)$ for k odd, and $S_k = \frac{2}{3}(2^k - 1)$ for k even. Let $P_k = a_1a_2 \cdots a_k$. It follows that, for k odd, P_k is an integer if and only if 19 divides $2^{k+1} - 1$; and, for k even, P_k is an integer if and only if 19 divides $2^k - 1$. A little computation shows that this first occurs at $k = 17$, when $2^{18} - 1 = 2^{18} - 1 = (2^9 - 1)(2^9 + 1) = 511 \cdot 513 = 511 \cdot 19 \cdot 27$. (In fact, one can show that P_k is an integer if and only if k is congruent to 0 or -1 mod 18.)

1999 28. **(E)** Let a, b, and c denote the number of -1's, 1's, and 2's in the sequence, respectively. We need not consider the zeros. Then a, b, c are nonnegative integers satisfying $-a + b + 2c = 19$ and $a + b + 4c = 99$. It follows that $a = 40 - c$ and $b = 59 - 3c$, where $0 \leq c \leq 19$ (since $b \geq 0$), so

$$x_1^3 + x_2^3 + \cdots + x_n^3 = -a + b + 8c = 19 + 6c.$$

The lower bound is achieved when $c = 0$ $(a = 40, b = 59)$. The upper bound is achieved when $c = 19$ $(a = 21, b = 2)$. Thus $m = 19$ and $M = 133$, so $M/m = 7$.