UNIT 13 EXERCISES 21-25

NUMBER THEO/ ALGEBRA WORD

2005A 21. **(C)** The two equations are equivalent to $b = a^{(c^{2005})}$ and $c = 2005 - b - a$, so

$$c = 2005 - a^{(c^{2005})} - a.$$

If $c > 1$, then

$$b \geq 2^{(2^{2005})} > 2005 > 2005 - a - c = b,$$

which is a contradiction. For $c = 0$ and for $c = 1$, the only solutions are the ordered triples $(2004, 1, 0)$ and $(1002, 1002, 1)$, respectively. Thus the number of solutions is 2.

2006A 21. **(E)** For $j = 1$ and 2, the given inequality is equivalent to

$$j + x^2 + y^2 \le 10^j(x + y),$$

or to

$$\left(x - \frac{10^j}{2}\right)^2 + \left(y - \frac{10^j}{2}\right)^2 \le \frac{10^{2j}}{2} - j,$$

provided that $x + y > 0$. These inequalities define regions bounded by circles. For $j = 1$ the circle has center $(5, 5)$ and radius 7. For $j = 2$ the circle has center $(50, 50)$ and radius $\sqrt{4998}$. In each case the center is on the line $y = x$ in the first quadrant, and the radius is less than the distance from the center to the origin. Thus $x + y > 0$ at each interior point of each circle, as was required to ensure the equivalence of the inequalities. The squares of the radii of the circles are 49 and 4998 for $j = 1$ and 2, respectively. Therefore the ratio of the area of S_2 to that of S_1 is $(4998\pi)/(49\pi) = 102$.

2007B 21. **Answer (A):** Because $3^6 = 729 < 2007 < 2187 = 3^7$, it is convenient to begin by counting the number of base-3 palindromes with at most 7 digits. There are two palindromes of length 1, namely 1 and 2. There are also two palindromes of length 2, namely 11 and 22. For $n \ge 1$, each palindrome of length $2n + 1$ is obtained by inserting one of the digits 0, 1, or 2 immediately after the nth digit in a palindrome of length $2n$. Each palindrome of length $2n+2$ is obtained by similarly inserting one of the strings 00, 11, or 22. Therefore there are 6 palindromes of each of the lengths 3 and 4, 18 of each of the lengths 5 and 6, and 54 of length 7. Because the base-3 representation of 2007 is 2202100, that integer is less than each of the palindromes 2210122, 2211122, 2212122, 2220222, 2221222, and 2222222. Thus the required total is $2+2+6+6+18+18+54-6 = 100$.

2017A

21. **Answer (D):** Because -1 is a root of $10x + 10$, -1 is added to S. Then 1 is also added to S, because it is a root of $(-1)x^{10} + (-1)x^9 + \cdots + (-1)x + 10$. At this point -10, a root of $1 \cdot x + 10$, can be added to S. Because 2 is a root of $1 \cdot x^3 + 0 \cdot x^2 + 1 \cdot x + (-10)$, and -2 is a root of $1 \cdot x + 2$, both 2 and -2 can be added to S. The polynomials $2x + (-10)$ and $2x + 10$ allow 5 and -5 into S. At this point $S = \{0, \pm 1, \pm 2, \pm 5, \pm 10\}$. No more elements can be added to S, because by the Rational Root Theorem, any integer root of a polynomial with integer coefficients whose constant term is a factor of 10 must be a factor of 10. Therefore S contains 9 elements.

Note: It is not true that in general if S starts with $\{0, c\}$ then all factors of c can be added to S. For example, applying the procedure to $\{0, 35\}$ gives only $\{0, \pm 1, \pm 35\}$, although of course it takes some argument to rule out ± 5 and ± 7.

2015A

22. **Answer (D):** Note that $S(1) = 2$, $S(2) = 4$, and $S(3) = 8$. Call a sequence with A and B entries valid if it does not contain 4 or more consecutive symbols that are the same. For $n \geq 4$, every valid sequence of length $n - 1$ can be extended to a valid sequence of length n by appending a symbol different from its last symbol. Similarly, valid sequences of length $n-2$ or $n-3$ can be extended to valid sequences of length n by appending either two or three equal symbols different from its last symbol. Note that all of these sequences are pairwise distinct. Conversely, every valid sequence of length n ends with either one, two, or three equal consecutive symbols. Removal of these equal symbols at the end produces every valid sequence of length $n-1$, $n-2$, or $n-3$, respectively. Thus $S(n) = S(n-1) + S(n-2) + S(n-3)$. This recursive formula implies that the remainders modulo 3 of the sequence $S(n)$ for $1 \leq n \leq 16$ are

$$2, 1, 2, 2, 2, 0, 1, 0, 1, 2, 0, 0, 2, 2, 1, 2.$$

Thus the sequence is periodic with period-length 13. Because $2015 = 13 \cdot 155$, it follows that $S(2015) \equiv S(13) \equiv 2 \pmod 3$. Similarly, the remainders modulo 4 of the sequence $S(n)$ for $1 \leq n \leq 7$ are $2, 0, 0, 2, 2, 0, 0$. Thus the sequence is periodic with period-length 4. Because $2015 = 4 \cdot 503 + 3$, it follows that $S(2015) \equiv S(3) \equiv 0 \pmod 4$. Therefore $S(2015) = 4k$ for some integer k, and $4k \equiv 2 \pmod 3$. Hence $k \equiv 2 \pmod 3$ and $S(2015) = 4k \equiv 8 \pmod{12}$.

2016B

22. **Answer (B):** Because $\frac{1}{n} = \frac{abcdef}{999999}$, it follows that n is a divisor of $10^6 - 1 = (10^3 - 1)(10^3 + 1) = 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$. Because $\frac{1}{n+6} = \frac{wxyz}{9999}$, it follows that $n + 6$ divides $10^4 - 1 = 3^2 \cdot 11 \cdot 101$. However, $n + 6$ does not divide $10^2 - 1 = 3^2 \cdot 11$, because otherwise the decimal representation of $\frac{1}{n+6}$ would have period 1 or 2. Thus $n = 101k - 6$, where $k = 1$, 3, 9, 11, 33, or 99. Because $n < 1000$, the only possible values of k are 1, 3, and 9, and the corresponding values of n are 95, 297, and 903. Of these, only $297 = 3^3 \cdot 11$ divides $10^6 - 1$. Thus $n \in [201, 400]$. It may be checked that $\frac{1}{297} = 0.\overline{003367}$ and $\frac{1}{303} = 0.\overline{0033}$.

2014A 23. **Answer (B):** Note that

$$\frac{10^n}{99^2} = \frac{10^n}{9801} = b_{n-1}b_{n-2}\dots b_2b_1b_0.\overline{b_{n-1}b_{n-2}\dots b_2b_1b_0}.$$

Subtracting the original equation gives

$$\frac{10^n - 1}{99^2} = b_{n-1}b_{n-2}\dots b_2b_1b_0.$$

Thus $10^n - 1 = 99^2 \cdot b_{n-1}b_{n-2}\dots b_2b_1b_0$. It follows that $10^n - 1$ is divisible by 11 and thus n is even, say $n = 2N$. For $0 \le j \le N-1$, let $a_j = 10b_{2j+1} + b_{2j}$. Note that $0 \le a_j \le 99$, and because

$$\frac{10^{2N}-1}{10^2-1} = 1 + 10^2 + 10^4 + \dots + 10^{2(N-1)},$$

it follows that

$$\sum_{k=0}^{N-1} 10^{2k} = (10^2 - 1)\sum_{k=0}^{N-1} a_k 10^{2k},$$

and so

$$\sum_{k=0}^{N-1} 10^{2k} + \sum_{k=0}^{N-1} a_k 10^{2k} = \sum_{k=1}^{N} a_{k-1} 10^{2k}.$$

Considering each side of the equation as numbers written in base 100, it follows that $1 + a_0 \equiv 0 \pmod{100}$, so $a_0 = 99$ and there is a carry for the 10^2 digit in the sum on the left side. Thus $1 + (1 + a_1) \equiv a_0 = 99 \pmod{100}$ and so $a_1 = 97$, and there is no carry for the 10^4 digit. Next, $1 + a_2 \equiv a_1 = 97 \pmod{100}$, and so $a_2 = 96$ with no carry for the 10^6 digit. In the same way $a_j = 98 - j$ for $1 \le j \le 98$. Then $1 + a_{99} \equiv a_{98} = 0 \pmod{100}$ would yield $a_{99} = 99$, and then the period would start again. Therefore $N = 99$ and $b_{n-1}b_{n-2}\dots b_2b_1b_0 = 0001020304\dots969799$. By momentarily including 9 and 8 as two extra digits, the sum would be $(0 + 1 + 2 + \dots + 9) \cdot 20 = 900$, so the required sum is $900 - 9 - 8 = 883$.

2012A

24. **Answer (C):**

Because $y = a^x$ is decreasing for $0 < a < 1$ and $y = x^b$ is increasing on the interval $[0, \infty)$ for $b > 0$, it follows that

$$1 > a_2 = (0.2011)^{a_1} > (0.201)^{a_1} > (0.201)^1 = a_1,$$
$$a_3 = (0.20101)^{a_2} < (0.2011)^{a_2} < (0.2011)^{a_1} = a_2,$$

and

$$a_3 = (0.20101)^{a_2} > (0.201)^{a_2} > (0.201)^1 = a_1.$$

Therefore $1 > a_2 > a_3 > a_1 > 0$. More generally, it can be shown by induction that

$$1 > b_1 = a_2 > b_2 = a_4 > \cdots > b_{1005} = a_{2010}$$

$$> b_{1006} = a_{2011} > b_{1007} = a_{2009} > \cdots > b_{2011} = a_1 > 0.$$

Hence $a_k = b_k$ if and only if $2(k - 1006) = 2011 - k$, so $k = 1341$.

1999

25. **(B)** Multiply both sides of the equation by 7! to obtain

$$3600 = 2520a_2 + 840a_3 + 210a_4 + 42a_5 + 7a_6 + a_7.$$

It follows that $3600 - a_7$ is a multiple of 7, which implies that $a_7 = 2$. Thus,

$$\frac{3598}{7} = 514 = 360a_2 + 120a_3 + 30a_4 + 6a_5 + a_6.$$

Reason as above to show that $514 - a_6$ is a multiple of 6, which implies that $a_6 = 4$. Thus, $510/6 = 85 = 60a_2 + 20a_3 + 5a_4 + a_5$. Then it follows that $85 - a_5$ is a multiple of 5, whence $a_5 = 0$. Continue in this fashion to obtain $a_4 = 1, a_3 = 1$, and $a_2 = 1$. Thus the desired sum is $1 + 1 + 1 + 0 + 4 + 2 = 9$.

2004B 25. **(B)** The smallest power of 2 with a given number of digits has a first digit of 1, and there are elements of S with n digits for each positive integer $n \le 603$, so there are 603 elements of S whose first digit is 1. Furthermore, if the first digit of 2^k is 1, then the first digit of 2^{k+1} is either 2 or 3, and the first digit of 2^{k+2} is either 4,5,6, or 7. Therefore there are 603 elements of S whose first digit is 2 or 3, 603 elements whose first digit is 4,5,6, or 7, and $2004-3(603) = 195$ whose first digit is 8 or 9. Finally, note that the first digit of 2^k is 8 or 9 if and only if the first digit of 2^{k-1} is 4, so there are 195 elements of S whose first digit is 4.

2006A 25. **(E)** For $1 \le k \le 15$, the k-element sets with properties (1) and (2) are the k-element subsets of $U_k = \{k, k+1, \ldots, 15\}$ that contain no two consecutive integers. If $\{a_1, a_2, \ldots, a_k\}$ is such a set, with its elements listed in increasing order, then $\{a_1 + k - 1, a_2 + k - 2, \ldots, a_{k-1} + 1, a_k\}$ is a k-element subset of U_{2k-1}. Conversely, if $\{b_1, b_2, \ldots, b_k\}$ is a k-element subset of U_{2k-1}, with its elements listed in increasing order, then $\{b_1 - k + 1, b_2 - k + 2, \ldots, b_{k-1} - 1, b_k\}$ is a set with properties (1) and (2). Thus for each k, the number of k-element sets with properties (1) and (2) is equal to the number of k-element subsets of the $(17-2k)$-element set U_{2k-1}. Because $k \le 17 - 2k$ only if $k \le 5$, the total number of such sets is

$$\sum_{k=1}^{5} \binom{17-2k}{k} = \binom{15}{1} + \binom{13}{2} + \binom{11}{3} + \binom{9}{4} + \binom{7}{5} = 15+78+165+126+21 = 405.$$

2016A

25. **Answer (E):** Assume that $k = 2j \geq 2$ is even. The smallest perfect square with $k+1$ digits is $10^k = (10^j)^2$. Thus the sequence of numbers written on the board after Silvia erases the last k digits of each number is the sequence

$$1 = \left\lfloor \frac{(10^j)^2}{10^k} \right\rfloor, \left\lfloor \frac{(10^j+1)^2}{10^k} \right\rfloor, \ldots, \left\lfloor \frac{n^2}{10^k} \right\rfloor, \ldots.$$

The sequence ends the first time that

$$\left\lfloor \frac{(n+1)^2}{10^k} \right\rfloor - \left\lfloor \frac{n^2}{10^k} \right\rfloor \geq 2;$$

before that, every two consecutive terms are either equal or they differ by 1. Suppose that

$$\left\lfloor \frac{n^2}{10^k} \right\rfloor = a \quad \text{and} \quad \left\lfloor \frac{(n+1)^2}{10^k} \right\rfloor \geq a+2.$$

Then $n^2 < (a+1)10^k$ and $(a+2)10^k \leq (n+1)^2$. Thus

$$10^k = (a+2)10^k - (a+1)10^k < (n+1)^2 - n^2 = 2n+1.$$

It follows that $n = \frac{10^k}{2} + m$ for some positive integer m. Note that

$$\frac{n^2}{10^k} = \frac{1}{10^k}\left(\frac{10^k}{2} + m\right)^2 = \frac{1}{10^k}\left(\frac{10^{2k}}{4} + m \cdot 10^k + m^2\right) = \frac{10^k}{4} + m + \frac{m^2}{10^k}.$$

Because $k \geq 2$, it follows that 10^k is divisible by 4, and so

$$\left\lfloor \frac{n^2}{10^k} \right\rfloor = \frac{10^k}{4} + m + \left\lfloor \frac{m^2}{10^k} \right\rfloor \quad \text{and} \quad \left\lfloor \frac{(n+1)^2}{10^k} \right\rfloor = \frac{10^k}{4} + m + 1 + \left\lfloor \frac{(m+1)^2}{10^k} \right\rfloor.$$

The difference will be at least 2 for the first time when

$$\left\lfloor \frac{m^2}{10^k} \right\rfloor = 0 \quad \text{and} \quad \left\lfloor \frac{(m+1)^2}{10^k} \right\rfloor \geq 1,$$

that is, for m such that $m^2 < 10^k \leq (m+1)^2$, equivalently, $m < 10^j \leq m+1$. Thus $m = 10^j - 1$ and then

$$f(k) = f(2j) = a + 1 = \left\lfloor \frac{n^2}{10^k} \right\rfloor + 1 = \frac{10^k}{4} + m + 1 = \frac{10^{2j}}{4} + 10^j.$$

Therefore

$$\sum_{j=1}^{1008} f(2j) = \sum_{j=1}^{1008}\left(\frac{10^{2j}}{4} + 10^j\right) = 25\sum_{j=0}^{1007} 10^{2j} + 10\sum_{j=0}^{1007} 10^j$$
$$= \underbrace{252525\ldots25}_{2016 \text{ digits}} + \underbrace{111\ldots10}_{1009 \text{ digits}}.$$

Because there are no carries in the sum, the required sum of digits equals $1008 \cdot (2+5) + 1008 \cdot 1 = 1008 \cdot 8 = 8064$.

2018A

25. **Answer (D):** The equation $C_n - B_n = A_n^2$ is equivalent to

$$c \cdot \frac{10^{2n}-1}{9} - b \cdot \frac{10^n - 1}{9} = a^2 \left(\frac{10^n - 1}{9}\right)^2.$$

Dividing by $10^n - 1$ and clearing fractions yields

$$(9c - a^2) \cdot 10^n = 9b - 9c - a^2.$$

As this must hold for two different values n_1 and n_2, there are two such equations, and subtracting them gives

$$\left(9c - a^2\right)\left(10^{n_1} - 10^{n_2}\right) = 0.$$

The second factor is non-zero, so $9c - a^2 = 0$ and thus $9b - 9c - a^2 = 0$. From this it follows that $c = \left(\frac{a}{3}\right)^2$ and $b = 2c$. Hence digit a must be 3, 6, or 9, with corresponding values 1, 4, or 9 for c, and 2, 8, or 18 for b. The case $b = 18$ is invalid, so there are just two triples of possible values for a, b, and c, namely $(3, 2, 1)$ and $(6, 8, 4)$. In fact, in these cases, $C_n - B_n = A_n^2$ for *all* positive integers n; for example, $4444 - 88 = 4356 = 66^2$. The second triple has the greater coordinate sum, $6 + 8 + 4 = 18$.

1999

30. **(D)** Let $m+n=s$. Then $m^3+n^3+3mn(m+n)=s^3$. Subtracting the given equation from the latter yields

$$s^3 - 33^3 = 3mns - 99mn.$$

It follows that $(s-33)(s^2+33s+33^2-3mn)=0$, hence either $s=33$ or $(m+n)^2+33(m+n)+33^2-3mn=0$. The second equation is equivalent to $(m-n)^2+(m+33)^2+(n+33)^2=0$, whose only solution, $(-33,-33)$, qualifies. On the other hand, the solutions to $m+n=33$ satisfying the required conditions are $(0,33),(1,32),(2,31),\ldots,(33,0)$, of which there are 34. Thus there are 35 solutions altogether.