UNIT 12 EXERCISES 21-25

SIMP FRACTIONS

2013A 21. **Answer (A):**

Let $A_n = \log\left(n + \log\left((n-1) + \log\left(\cdots + \log\left(3 + \log 2\right)\cdots\right)\right)\right)$. Note that $0 < \log 2 = A_2 < 1$. If $0 < A_{k-1} < 1$, then $k < k + A_{k-1} < k+1$. Hence $0 < \log k < \log(k + A_{k-1}) = A_k < \log(k+1) \leq 1$, as long as $\log k > 0$ and $\log(k+1) \leq 1$, which occurs when $2 \leq k \leq 9$. Thus $0 < A_n < 1$ for $2 \leq n \leq 9$.

Because $0 < A_9 < 1$, it follows that $10 < 10 + A_9 < 11$, and so $1 = \log(10) < \log(10 + A_9) = A_{10} < \log(11) < 2$. If $1 < A_{k-1} < 2$, then $k+1 < k + A_{k-1} < k+2$. Hence $1 < \log(k+1) < \log(k + A_{k-1}) = A_k < \log(k+2) \leq 2$, as long as $\log(k+1) > 1$ and $\log(k+2) \leq 2$, which occurs when $10 \leq k \leq 98$. Thus $1 < A_n < 2$ for $10 \leq n \leq 98$.

In a similar way, it can be proved that $2 < A_n < 3$ for $99 \leq n \leq 997$, and $3 < A_n < 4$ for $998 \leq n \leq 9996$.

For $n = 2012$, it follows that $3 < A_{2012} < 4$, so $2016 < 2013 + A_{2012} < 2017$ and $\log 2016 < A_{2013} < \log 2017$.