UNIT 11 EXERCISES 21-25

DIVISOR/LCM

2005B

21. **(C)** Let $n = 7^k Q$, where Q is the product of primes, none of which is 7. Let d be the number of divisors of Q. Then n has $(k+1)d$ divisors. Also $7n = 7^{k+1}Q$, so $7n$ has $(k+2)d$ divisors. Thus

$$\frac{(k+2)d}{(k+1)d} = \frac{80}{60} = \frac{4}{3} \quad \text{and} \quad 3(k+2) = 4(k+1).$$

Hence $k = 2$. Note that $n = 2^{19}7^2$ meets the conditions of the problem.

2016A 22. **Answer (A):** Because $\text{lcm}(x,y) = 2^3 \cdot 3^2$ and $\text{lcm}(x,z) = 2^3 \cdot 3 \cdot 5^2$, it follows that 5^2 divides z, but neither x nor y is divisible by 5. Furthermore, y is divisible by 3^2, and neither x nor z is divisible by 3^2, but at least one of x or z is divisible by 3. Finally, because $\text{lcm}(y,z) = 2^2 \cdot 3^2 \cdot 5^2$, at least one of y or z is divisible by 2^2, but neither is divisible by 2^3. However, x must be divisible by 2^3. Thus $x = 2^3 \cdot 3^j$, $y = 2^k \cdot 3^2$, and $z = 2^m \cdot 3^n \cdot 5^2$, where $\max(j,n) = 1$ and $\max(k,m) = 2$. There are 3 choices for (j,n) and 5 choices for (k,m), so there are 15 possible ordered triples (x,y,z).

2014B 23. **Answer (C):** Let $n = \binom{2014}{k}$. Note that $2016 \cdot 2015 \equiv (-1)(-2) = 2 \pmod{2017}$ and $2016 \cdot 2015 \cdots (2015-k) \equiv (-1)(-2)\cdots(-(k+2)) = (-1)^k(k+2)! \pmod{2017}$. Because $n \cdot k! \cdot (2014-k)! = 2014!$, it follows that

$$n \cdot k! \cdot (2014-k)! \cdot ((2015-k)\cdots 2015 \cdot 2016) \cdot 2 \equiv 2014! \cdot 2015 \cdot 2016 \cdot (-1)^k(k+2)! \pmod{2017}.$$

Thus

$$2n \cdot k! \cdot 2016! \equiv (-1)^k(k+2)! \cdot 2016! \pmod{2017}.$$

Dividing by $2016! \cdot k!$, which is relatively prime to 2017, gives

$$2n \equiv (-1)^k(k+2)(k+1) \pmod{2017}.$$

Thus $n \equiv (-1)^k\binom{k+2}{2} \pmod{2017}$. It follows that

$$S \equiv \sum_{k=0}^{62}(-1)^k\binom{k+2}{2} = 1 + \sum_{k=1}^{31}\left(\binom{2k+2}{2} - \binom{2k+1}{2}\right)$$
$$= 1 + \sum_{k=1}^{31}(2k+1) = 32^2 = 1024 \pmod{2017}.$$

2016B

24. **Answer (D):** Note that $\gcd(a,b,c,d) = 77$ and $\operatorname{lcm}(a,b,c,d) = n$ if and only if $\gcd(\frac{a}{77}, \frac{b}{77}, \frac{c}{77}, \frac{d}{77}) = 1$ and $\operatorname{lcm}(\frac{a}{77}, \frac{b}{77}, \frac{c}{77}, \frac{d}{77}) = \frac{n}{77}$. Thus there are 77,000 ordered quadruples (a,b,c,d) such that $\gcd(a,b,c,d) = 1$ and $\operatorname{lcm}(a,b,c,d) = \frac{n}{77}$. Let $m = \frac{n}{77}$ and suppose that p is a prime that divides m. Let $A = A(p)$, $B = B(p)$, $C = C(p)$, $D = D(p)$, and $M = M(p) \geq 1$ be the exponents of p such that p^A, p^B, p^C, p^D, and p^M are the largest powers of p that divide a, b, c, d, and m, respectively. The gcd and lcm requirements are equivalent to $\min(A,B,C,D) = 0$ and $\max(A,B,C,D) = M$. For a fixed value of M, there are $(M+1)^4$ quadruples (A,B,C,D) with each entry in $\{0,1,\ldots,M\}$. There are M^4 of them for which $\min(A,B,C,D) \geq 1$, and also M^4 of them such that $\max(A,B,C,D) \leq M-1$. Finally, there are $(M-1)^4$ quadruples (A,B,C,D) such that $\min(A,B,C,D) \geq 1$ and $\max(A,B,C,D) \leq M-1$. Thus the number of quadruples such that $\min(A,B,C,D) = 0$ and $\max(A,B,C,D) = M$ is equal to $(M+1)^4 - 2M^4 + (M-1)^4 = 12M^2 + 2 = 2(6M^2+1)$. Multiplying these quantities over all primes that divide m yields the total number of quadruples (a,b,c,d) with the required properties. Thus

$$77{,}000 = 2^3 \cdot 5^3 \cdot 7 \cdot 11 = \prod_{p|m} 2(6(M(p))^2 + 1).$$

Note that $6(M(p))^2 + 1$ is odd and this product must contain three factors of 2, so there must be exactly three primes that divide m. Let p_1, p_2, and p_3 be these primes. Note that $6 \cdot 1^2 + 1 = 7$, $6 \cdot 2^2 + 1 = 5^2$, and $6 \cdot 3^2 + 1 = 5 \cdot 11$. None of these could appear as a factor more than once because 77,000 is not divisible by 7^2, 5^4, or 11^2. Moreover, the product of these three is equal to $5^3 \cdot 7 \cdot 11$. All other factors of the form $6M^2 + 1$ are greater than these three, so without loss of generality the only solution is $M(p_1) = 1$, $M(p_2) = 2$, and $M(p_3) = 3$. It follows that $m = p_1^1 p_2^2 p_3^3$, and the smallest value of m occurs when $p_1 = 5$, $p_2 = 3$, and $p_3 = 2$. Therefore the smallest possible values of m and n are $5 \cdot 3^2 \cdot 2^3 = 360$ and $77(5 \cdot 3^2 \cdot 2^3) = 27{,}720$, respectively.