UNIT 10 EXERCISES 21-25

ALGEBRA

2002B 22. **(B)** We have $a_n = \frac{1}{\log_n 2002} = \log_{2002} n$, so

$$\begin{aligned} b - c &= (\log_{2002} 2 + \log_{2002} 3 + \log_{2002} 4 + \log_{2002} 5) \\ &\quad - (\log_{2002} 10 + \log_{2002} 11 + \log_{2002} 12 + \log_{2002} 13 + \log_{2002} 14) \\ &= \log_{2002} \frac{2 \cdot 3 \cdot 4 \cdot 5}{10 \cdot 11 \cdot 12 \cdot 13 \cdot 14} = \log_{2002} \frac{1}{11 \cdot 13 \cdot 14} = \log_{2002} \frac{1}{2002} = -1. \end{aligned}$$

2006B 22. **(B)** Note that n is the number of factors of 5 in the product $a!b!c!$, and $2006 < 5^5$. Thus

$$n = \sum_{k=1}^{4} \left(\lfloor a/5^k \rfloor + \lfloor b/5^k \rfloor + \lfloor c/5^k \rfloor \right).$$

Because $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor z \rfloor \geq \lfloor x + y + z \rfloor - 2$ for all real numbers x, y, and z, it follows that

$$\begin{aligned} n &\geq \sum_{k=1}^{4} \left(\lfloor (a+b+c)/5^k \rfloor - 2 \right) \\ &= \sum_{k=1}^{4} \left(\lfloor 2006/5^k \rfloor - 2 \right) \\ &= 401 + 80 + 16 + 3 - 4 \cdot 2 = 492. \end{aligned}$$

The minimum value of 492 is achieved, for example, when $a = b = 624$ and $c = 758$.

2010B 22. **Answer (D):** Let R be the circumradius of $ABCD$ and let $a = AB$, $b = BC$, $c = CD$, $d = DA$, and $k = bc = ad$. Because the areas of $\triangle ABC$, $\triangle CDA$, $\triangle BCD$, and $\triangle ABD$ are

$$\frac{ab \cdot AC}{4R}, \quad \frac{cd \cdot AC}{4R}, \quad \frac{bc \cdot BD}{4R}, \quad \text{and} \quad \frac{ad \cdot BD}{4R},$$

respectively, and Area($\triangle ABC$)+Area($\triangle CDA$) = Area($\triangle BCD$)+Area($\triangle ABD$), it follows that

$$\frac{AC}{4R}(ab + cd) = \frac{BD}{4R}(bc + ad) = \frac{BD}{4R}(2k);$$

that is, $(ab + cd) \cdot AC = 2k \cdot BD$. By Ptolemy's Theorem $ac + bd = AC \cdot BD$. Solving for AC and substituting into the previous equation gives

$$BD^2 = \frac{1}{2k}(ac+bd)(ab+cd) = \frac{1}{2k}(a^2k+c^2k+b^2k+d^2k) = \frac{1}{2}(a^2+b^2+c^2+d^2).$$

None of the sides can be equal to 11 or 13 because by assumption $a, b, c,$ and d are pairwise distinct and less than 15, and so it is impossible to have a factor of 11 or 13 on each side of the equation $bc = ad$. If the largest side length is 12 or less, then $2BD^2 \le 12^2 + 10^2 + 9^2 + 8^2 = 389$, and so $BD \le \sqrt{\frac{389}{2}}$. If the largest side is 14 and the other sides are $s_1 > s_2 > s_3$, then $14s_3 = s_1s_2$. Thus 7 divides s_1s_2 and because $0 < s_2 < s_1 < 14$, it follows that either $s_1 = 7$ or $s_2 = 7$. If $s_1 = 7$, then $2BD^2 < 14^2 + 7^2 + 6^2 + 5^2 = 306$. If $s_2 = 7$, then $2s_3 = s_1$, and it follows that $2BD^2 \le 14^2 + 7^2 + 12^2 + 6^2 = 425$. Therefore $BD \le \sqrt{\frac{425}{2}}$ with equality for a cyclic quadrilateral with $a = 14$, $b = 12$, $c = 7$, and $d = 6$.

2012A 22. **Answer (C):** Label the vertices of Q as in the figure. Let m_{xy} denote the midpoint of $\overline{v_x v_y}$. Call a segment *long* if it joins midpoints of opposite edges of a face and *short* if it joins midpoints of adjacent edges.

Let p be one of the k planes. Assume p intersects the face $v_1v_2v_3v_4$. First suppose p intersects $v_1v_2v_3v_4$ by a long segment. By symmetry assume $p \cap v_1v_2v_3v_4 = \overline{m_{12}m_{34}}$. Because p intersects the interior of Q, it follows that p intersects the face $v_3v_4v_8v_7$. By symmetry there are two cases: 1.1 $p \cap v_3v_4v_8v_7 = \overline{m_{34}m_{78}}$ and 1.2 $p \cap v_3v_4v_8v_7 = \overline{m_{34}m_{48}}$.

In Case 1.1 the plane p is the plane determined by the square $m_{12}m_{34}m_{78}m_{56}$. Note that p contains 4 long segments and by symmetry there are 3 planes like p, one for every pair of opposite faces of Q.

In Case 1.2 the plane p is determined by the rectangle $m_{12}m_{34}m_{48}m_{15}$. Note that p contains 2 long segments and 2 short segments, and by symmetry there are 12 planes like p, one for every edge of Q.

Second, suppose p intersects $v_1v_2v_3v_4$ by a short segment. By symmetry assume $p \cap v_1v_2v_3v_4 = \overline{m_{23}m_{34}}$. Again p must intersect the face $v_3v_4v_8v_7$. There are three cases: 2.1 $p \cap v_3v_4v_8v_7 = \overline{m_{34}m_{37}}$, 2.2 $p \cap v_3v_4v_8v_7 = \overline{m_{34}m_{78}}$, and 2.3 $p \cap v_3v_4v_8v_7 = \overline{m_{34}m_{48}}$.

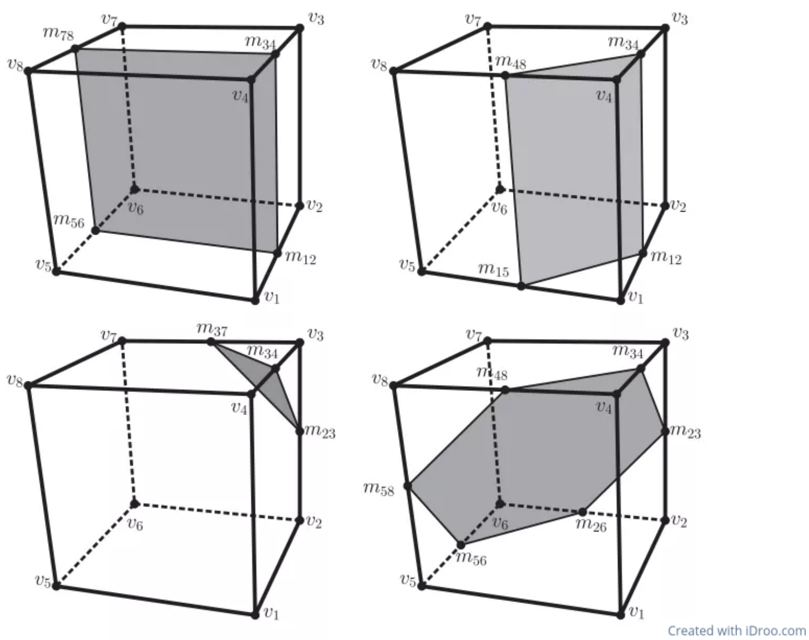
In Case 2.1 the plane p is the plane determined by the triangle $m_{23}m_{34}m_{37}$. Note that p contains 3 short segments and by symmetry there are 8 planes like p, one for every vertex of Q.

Case 2.2 duplicates Case 1.2.

In Case 2.3 the plane p is determined by the hexagon $m_{23}m_{34}m_{48}m_{58}m_{56}m_{26}$. Note that p contains 6 short segments, and by symmetry there are 4 planes like p, one for every pair of opposite vertices of Q.

Therefore the maximum possible value of k is $3 + 12 + 8 + 4 = 27$, obtained by considering all possible planes classified so far.

To find the minimum, note that $P \cap S$ consists of 24 short segments and 12 long segments. Every plane $p \in P$ can contain at most 6 short segments; moreover, the union of the 4 planes obtained from Case 2.3 contains all 24 short segments. Similarly, every plane $p \in P$ can contain at most 4 long segments; moreover, the union of the 3 planes obtained from Case 1.1 contains all 12 long segments. Thus the minimum possible value of k is $4 + 3 = 7$, and the required difference is $27 - 7 = 20$.



2014A

22. **Answer (B):** Because $2^2 < 5$ and $2^3 > 5$, there are either two or three integer powers of 2 strictly between any two consecutive integer powers of 5. Thus for each n there is at most one m satisfying the given inequalities, and the question asks for the number of cases in which there are three powers rather than two. Let d (respectively, t) be the number of nonnegative integers n less than 867 such that there are exactly two (respectively, three) powers of 2 strictly between 5^n and 5^{n+1}. Because $2^{2013} < 5^{867} < 2^{2014}$, it follows that $d + t = 867$ and $2d + 3t = 2013$. Solving the system yields $t = 279$.

2003A

23. (B) We have

$$\begin{aligned}1! \cdot 2! \cdot 3! \cdots 9! =&(1)(1 \cdot 2)(1 \cdot 2 \cdot 3) \cdots (1 \cdot 2 \cdots 9)\\ =&1^9 2^8 3^7 4^6 5^5 6^4 7^3 8^2 9^1 = 2^{30} 3^{13} 5^5 7^3.\end{aligned}$$

The perfect square divisors of that product are the numbers of the form

$$2^{2a} 3^{2b} 5^{2c} 7^{2d}$$

with $0 \le a \le 15$, $0 \le b \le 6$, $0 \le c \le 2$, and $0 \le d \le 1$. Thus there are $(16)(7)(3)(2) = 672$ such numbers.

2004A 23. **(E)** Since $z_1 = 0$, it follows that $c_0 = P(0) = 0$. The nonreal zeros of P must occur in conjugate pairs, so $\sum_{k=1}^{2004} b_k = 0$ and $\sum_{k=1}^{2004} a_k = 0$ also. The coefficient c_{2003} is the sum of the zeros of P, which is

$$\sum_{k=1}^{2004} z_k = \sum_{k=1}^{2004} a_k + i\sum_{k=1}^{2004} b_k = 0.$$

Finally, since the degree of P is even, at least one of $z_2, \ldots, z_{2004}$ must be real, so at least one of $b_2, \ldots, b_{2004}$ is 0 and consequently $b_2 \cdot b_3 \cdot \ldots \cdot b_{2004} = 0$. Thus the quantities in **(A)**, **(B)**, **(C)**, and **(D)** must all be 0.

Note that the polynomial

$$P(x) = x(x-2)(x-3)\cdots(x-2003)\left(x + \sum_{k=2}^{2003} k\right)$$

satisfies the given conditions, and $\sum_{k=1}^{2004} c_k = P(1) \neq 0$.

2008B 23. **Answer (A):** Because the prime factorization of 10 is $2 \cdot 5$, the positive divisors of 10^n are the numbers $2^a \cdot 5^b$ with $0 \le a \le n$ and $0 \le b \le n$. Thus

$$\begin{aligned}
792 = \sum_{a=0}^{n}\sum_{b=0}^{n}\log_{10}\left(2^a 5^b\right) &= \sum_{a=0}^{n}\sum_{b=0}^{n}\left(a\log_{10}2 + b\log_{10}5\right) \\
&= \sum_{b=0}^{n}\sum_{a=0}^{n}\left(a\log_{10}2\right) + \sum_{a=0}^{n}\sum_{b=0}^{n}\left(b\log_{10}5\right) \\
&= (n+1)\left(\log_{10}2\right)\sum_{a=0}^{n}a + (n+1)\left(\log_{10}5\right)\sum_{b=0}^{n}b \\
&= (n+1)\left(\log_{10}2 + \log_{10}5\right)\left(\frac{1}{2}n(n+1)\right) \\
&= \frac{1}{2}n(n+1)^2\left(\log_{10}10\right) = \frac{1}{2}n(n+1)^2.
\end{aligned}$$

Hence $n(n+1)^2 = 2 \cdot 792 = 2 \cdot 11 \cdot 72 = 11 \cdot 12^2$, so $n = 11$.

OR

Let $d(M)$ denote the number of divisors of a positive integer M. The sum of the logs of the divisors of M is equal to the log of the product of its divisors.

If M is not a square, its divisors can be arranged in pairs, each with a product of M. Thus the product of the divisors is $M^{d(M)/2}$. A similar argument shows that this result is also true if M is a square. Therefore

$$792 = \log\left(\left(10^n\right)^{d(10^n)/2}\right) = \frac{1}{2}d(10^n) \cdot n = \frac{1}{2}d(2^n \cdot 5^n) \cdot n = \frac{1}{2}(n+1)^2 \cdot n,$$

and the conclusion follows as in the first solution.

2010A 23. **Answer (A):** There are 18 factors of 90! that are multiples of 5, 3 factors that are multiples of 25, and no factors that are multiples of higher powers of 5. Also, there are more than 45 factors of 2 in 90!. Thus $90! = 10^{21}N$ where N is an integer not divisible by 10, and if $N \equiv n \pmod{100}$ with $0 < n \le 99$, then n is a multiple of 4.

Let $90! = AB$ where A consists of the factors that are relatively prime to 5 and B consists of the factors that are divisible by 5. Note that $\prod_{j=1}^{4}(5k+j) \equiv 5k(1+2+3+4)+1\cdot 2\cdot 3\cdot 4 \equiv 24 \pmod{25}$, thus

$$\begin{aligned} A &= (1\cdot 2\cdot 3\cdot 4)\cdot(6\cdot 7\cdot 8\cdot 9)\cdot\cdots\cdot(86\cdot 87\cdot 88\cdot 89) \\ &\equiv 24^{18} \equiv (-1)^{18} \equiv 1 \pmod{25}. \end{aligned}$$

Similarly,

$$B = (5\cdot 10\cdot 15\cdot 20)\cdot(30\cdot 35\cdot 40\cdot 45)\cdot(55\cdot 60\cdot 65\cdot 70)\cdot(80\cdot 85\cdot 90)\cdot(25\cdot 50\cdot 75),$$

thus

$$\begin{aligned} \frac{B}{5^{21}} &= (1\cdot 2\cdot 3\cdot 4)\cdot(6\cdot 7\cdot 8\cdot 9)\cdot(11\cdot 12\cdot 13\cdot 14)\cdot(16\cdot 17\cdot 18)\cdot(1\cdot 2\cdot 3) \\ &\equiv 24^3\cdot(-9)\cdot(-8)\cdot(-7)\cdot 6 \equiv (-1)^3\cdot 1 \equiv -1 \pmod{25}. \end{aligned}$$

Finally, $2^{21} = 2\cdot(2^{10})^2 = 2\cdot(1024)^2 \equiv 2\cdot(-1)^2 \equiv 2 \pmod{25}$, so $13\cdot 2^{21} \equiv 13\cdot 2 \equiv 1 \pmod{25}$. Therefore

$$\begin{aligned} N &\equiv (13\cdot 2^{21})N = 13\cdot\frac{90!}{5^{21}} = 13\cdot A\cdot\frac{B}{5^{21}} \equiv 13\cdot 1\cdot(-1) \pmod{25} \\ &\equiv -13 \equiv 12 \pmod{25}. \end{aligned}$$

Thus n is equal to 12, 37, 62, or 87, and because n is a multiple of 4, it follows that $n = 12$.

2003A 24. (B) We have

$$\begin{aligned}\log_a \frac{a}{b} + \log_b \frac{b}{a} =& \log_a a - \log_a b + \log_b b - \log_b a \\ =& 1 - \log_a b + 1 - \log_b a \\ =& 2 - \log_a b - \log_b a.\end{aligned}$$

Let $c = \log_a b$, and note that $c > 0$ since a and b are both greater than 1. Thus

$$\log_a \frac{a}{b} + \log_b \frac{b}{a} = 2 - c - \frac{1}{c} = \frac{c^2 - 2c + 1}{-c} = \frac{(c-1)^2}{-c} \leq 0.$$

This expression is 0 when $c = 1$, that is, when $a = b$.

OR

As above

$$\log_a \frac{a}{b} + \log_b \frac{b}{a} = 2 - c - \frac{1}{c}$$

From the Arithmetic-Geometric Mean Inequality we have

$$\frac{c + 1/c}{2} \geq \sqrt{c \cdot \frac{1}{c}} = 1, \quad \text{so} \quad c + \frac{1}{c} \geq 2$$

and

$$\log_a \frac{a}{b} + \log_b \frac{b}{a} = 2 - \left(c + \frac{1}{c}\right) \leq 0$$

with equality when $c = \frac{1}{c}$, that is, when $a = b$.

2006A 24. **(D)** There is exactly one term in the simplified expression for every monomial of the form $x^ay^bz^c$, where a, b, and c are non-negative integers, a is even, and $a + b + c = 2006$. There are 1004 even values of a with $0 \le a \le 2006$. For each such value, b can assume any of the $2007 - a$ integer values between 0 and $2006-a$, inclusive, and the value of c is then uniquely determined as $2006-a-b$. Thus the number of terms in the simplified expression is

$$(2007 - 0) + (2007 - 2) + \cdots + (2007 - 2006) = 2007 + 2005 + \cdots + 1.$$

This is the sum of the first 1004 odd positive integers, which is $1004^2 = 1{,}008{,}016$.

OR

The given expression is equal to

$$\sum \frac{2006!}{a!b!c!} \left(x^ay^bz^c + x^a(-y)^b(-z)^c\right),$$

where the sum is taken over all non-negative integers a, b, and c with $a+b+c = 2006$. Because the number of non-negative integer solutions of $a + b + c = k$ is $\binom{k+2}{2}$, the sum is taken over $\binom{2008}{2}$ terms, but those for which b and c have opposite parity have a sum of zero. If b is odd and c is even, then a is odd, so $a = 2A+1, b = 2B+1$, and $c = 2C$ for some non-negative integers A, B, and C. Therefore $2A + 1 + 2B + 1 + 2C = 2006$, so $A + B + C = 1002$. Because the last equation has $\binom{1004}{2}$ non-negative integer solutions, there are $\binom{1004}{2}$ terms for which b is odd and c is even. The number of terms for which b is even and c is odd is the same. Thus the number of terms in the simplified expression is

$$\binom{2008}{2} - 2\binom{1004}{2} = 1004 \cdot 2007 - 1004 \cdot 1003 = 1004^2 = 1{,}008{,}016.$$

2007B 24. **Answer (A):** Let $u = a/b$. Then the problem is equivalent to finding all positive rational numbers u such that

$$u + \frac{14}{9u} = k$$

for some integer k. This equation is equivalent to $9u^2 - 9uk + 14 = 0$, whose solutions are

$$u = \frac{9k \pm \sqrt{81k^2 - 504}}{18} = \frac{k}{2} \pm \frac{1}{6}\sqrt{9k^2 - 56}.$$

Hence u is rational if and only if $\sqrt{9k^2 - 56}$ is rational, which is true if and only if $9k^2 - 56$ is a perfect square. Suppose that $9k^2 - 56 = s^2$ for some positive integer s. Then $(3k - s)(3k + s) = 56$. The only factors of 56 are 1, 2, 4, 7, 8, 14, 28, and 56, so $(3k - s, 3k + s)$ is one of the ordered pairs $(1, 56)$, $(2, 28)$, $(4, 14)$, or $(7, 8)$. The cases $(1, 56)$ and $(7, 8)$ yield no integer solutions. The cases $(2, 28)$ and $(4, 14)$ yield $k = 5$ and $k = 3$, respectively. If $k = 5$, then $u = 1/3$ or $u = 14/3$. If $k = 3$, then $u = 2/3$ or $u = 7/3$. Therefore there are four pairs (a, b) that satisfy the given conditions, namely $(1, 3)$, $(2, 3)$, $(7, 3)$, and $(14, 3)$.

Rewrite the equation

$$\frac{a}{b} + \frac{14b}{9a} = k$$

in two different forms. First, multiply both sides by b and subtract a to obtain

$$\frac{14b^2}{9a} = bk - a.$$

Because a, b, and k are integers, $14b^2$ must be a multiple of a, and because a and b have no common factors greater than 1, it follows that 14 is divisible by a. Next, multiply both sides of the original equation by $9a$ and subtract $14b$ to obtain

$$\frac{9a^2}{b} = 9ak - 14b.$$

This shows that $9a^2$ is a multiple of b, so 9 must be divisible by b. Thus if (a, b) is a solution, then $b = 1$, 3, or 9, and $a = 1$, 2, 7, or 14. This gives a total of twelve possible solutions (a, b), each of which can be checked quickly. The only such pairs for which

$$\frac{a}{b} + \frac{14b}{9a}$$

is an integer are when (a, b) is $(1, 3)$, $(2, 3)$, $(7, 3)$, or $(14, 3)$.

2010B

24. **Answer (C):** Let $f(x) = \frac{1}{x-2009} + \frac{1}{x-2010} + \frac{1}{x-2011}$. Note that

$$f(x) - f(y) = (y - x)\left(\frac{1}{(x-2009)(y-2009)} + \frac{1}{(x-2010)(y-2010)} + \frac{1}{(x-2011)(y-2011)}\right).$$

If $x < y < 2009$, then $y - x > 0$,

$$\frac{1}{(x-2009)(y-2009)} > 0, \quad \frac{1}{(x-2010)(y-2010)} > 0,$$
$$\text{and } \frac{1}{(x-2011)(y-2011)} > 0.$$

Thus f is decreasing on the interval $x < 2009$, and because $f(x) < 0$ for $x < 0$, it follows that no values $x < 2009$ satisfy $f(x) \geq 1$.

If $2009 < x < y < 2010$, then $f(x) - f(y) > 0$ as before. Thus f is decreasing in the interval $2009 < x < 2010$. Moreover, $f(2009 + \frac{1}{10}) = 10 - \frac{10}{9} - \frac{10}{19} > 1$ and $f(2010 - \frac{1}{10}) = \frac{10}{9} - 10 - \frac{10}{11} < 1$. Thus there is a number $2009 < x_1 < 2010$ such that $f(x) \geq 1$ for $2009 < x \leq x_1$ and $f(x) < 1$ for $x_1 < x < 2010$.

Similarly, f is decreasing on the interval $2010 < x < 2011$, $f(2010 + \frac{1}{10}) > 1$, and $f(2011 - \frac{1}{10}) < 1$. Thus there is a number $2010 < x_2 < 2011$ such that $f(x) \geq 1$ for $2010 < x \leq x_2$ and $f(x) < 1$ for $x_2 < x < 2011$.

Finally, f is decreasing on the interval $x > 2011$, $f(2011 + \frac{1}{10}) > 1$, and $f(2014) = \frac{1}{5} + \frac{1}{4} + \frac{1}{3} < 1$. Thus there is a number $x_3 > 2011$ such that $f(x) \geq 1$ for $2011 < x \leq x_3$ and $f(x) < 1$ for $x > x_3$.

The required sum of the lengths of these three intervals is

$$x_1 - 2009 + x_2 - 2010 + x_3 - 2011 = x_1 + x_2 + x_3 - 6020.$$

Multiplying both sides of the equation

$$\frac{1}{x-2009} + \frac{1}{x-2010} + \frac{1}{x-2011} = 1$$

by $(x-2009)(x-2010)(x-2011)$ and collecting terms on one side of the equation gives

$$x^3 - x^2(2009 + 2010 + 2011 + 1 + 1 + 1) + ax + b = 0$$

where a and b are real numbers. The three roots of this equation are x_1, x_2, and x_3. Thus $x_1 + x_2 + x_3 = 6020 + 3$, and consequently the required sum equals 3.

2018B 24. **Answer (C):** Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. Then $0 \le \{x\} < 1$. The given equation is equivalent to $x^2 = 10{,}000\{x\}$, that is,

$$\frac{x^2}{10{,}000} = \{x\}.$$

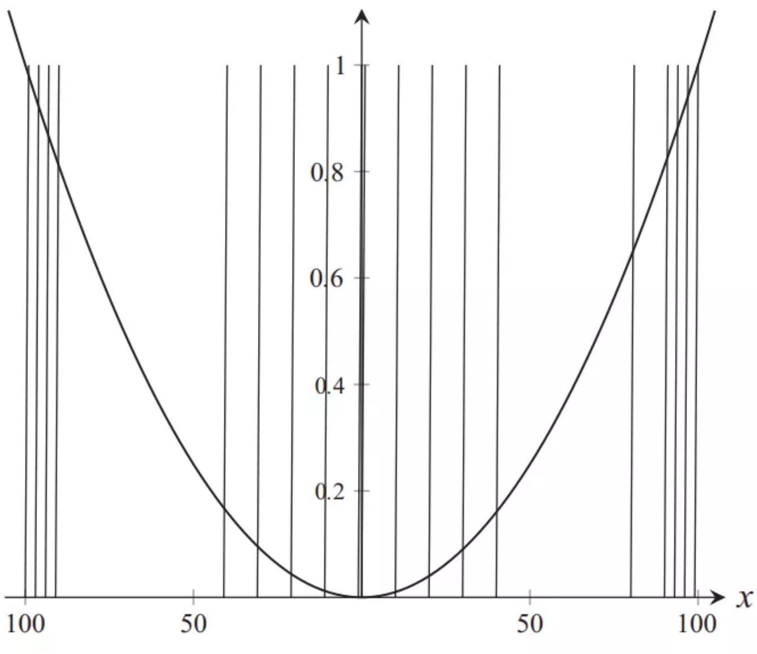
Therefore if x satisfies the equation, then

$$0 \le \frac{x^2}{10{,}000} < 1.$$

This implies that $x^2 < 10{,}000$, so $-100 < x < 100$. The figure shows a sketch of the graphs of

$$f(x) = \frac{x^2}{10{,}000} \qquad \text{and} \qquad g(x) = \{x\}$$

for $-100 < x < 100$ on the same coordinate axes. The graph of g consists of the 200 half-open line segments with slope 1 connecting the points $(k, 0)$ and $(k+1, 1)$ for $k = -100, -99, \ldots, 98, 99$. (The endpoints of these intervals that lie on the x-axis are part of the graph, but the endpoints with y-coordinate 1 are not.) It is clear that there is one intersection point for x lying in each of the intervals $[-100, -99)$, $[-99, -98)$, $[-98, -97)$, $\ldots$, $[-1, 0)$, $[0, 1)$, $[1, 2)$, $\ldots$, $[97, 98)$, $[98, 99)$ but no others. Thus the equation has 199 solutions.



OR

The solutions to the equation correspond to points of intersection of the graphs $y = 10000\lfloor x \rfloor$ and $y = 10000x - x^2$. There will be a point of intersection any time the parabola intersects the half-open horizontal segment from the point $(a, 10000a)$ to the point $(a+1, 10000a)$, where a is an integer. This occurs for every integer value of a for which

$$10000a - a^2 \le 10000a < 10000(a+1) - (a+1)^2.$$

This is equivalent to $(a+1)^2 < 10000$, which occurs if and only if $-101 < a < 99$. Thus points of intersection occur on the intervals $[a, a+1)$ for $a = -100, -99, -98, \ldots, -1, 0, 1, \ldots, 97, 98$, resulting in

2004A 25. **(E)** Note that $n^3 a_n = 133.\overline{133}_n = a_n + n^2 + 3n + 3$, so

$$a_n = \frac{n^2+3n+3}{n^3-1} = \frac{(n+1)^3-1}{n(n^3-1)}.$$

Therefore

$$\begin{aligned} a_4 a_5 \cdots a_{99} &= \left(\frac{5^3-1}{4(4^3-1)}\right)\left(\frac{6^3-1}{5(5^3-1)}\right)\cdots\left(\frac{100^3-1}{99(99^3-1)}\right) \\ &= \left(\frac{3!}{99!}\right)\left(\frac{100^3-1}{4^3-1}\right) = \left(\frac{6}{99!}\right)\left(\frac{99(100^2+100+1)}{63}\right) \\ &= \frac{(2)(10{,}101)}{(21)(98!)} = \frac{962}{98!}. \end{aligned}$$

2010B

25. **Answer (D):** Observe that $2010 = 2 \cdot 3 \cdot 5 \cdot 67$. Let $P = \prod_{n=2}^{5300} \text{pow}(n) = 2^a \cdot 3^b \cdot 5^c \cdot 67^d \cdot Q$ where Q is relatively prime with $2, 3, 5$, and 67. The largest power of 2010 that divides P is equal to 2010^m where $m = \min(a, b, c, d)$.

By definition $\text{pow}(n) = 2^k$ if and only if $n = 2^k$. Because $2^{12} = 4096 < 5300 < 8192 = 2^{13}$, it follows that

$$a = 1 + 2 + \cdots + 12 = \frac{12 \cdot 13}{2} = 78.$$

Similarly, $\text{pow}(n) = 67$ if and only if $n = 67N$ and the largest prime dividing N is smaller than 67. Because $5300 = 79 \cdot 67 + 7$ and 71, 73, and 79 are the only primes p in the range $67 < p \leq 79$; it follows that for $n \leq 5300$, $\text{pow}(n) = 67$ if and only if

$$n \in \{67k : 1 \leq k \leq 79\} \setminus \{67^2, 67 \cdot 71, 67 \cdot 73, 67 \cdot 79\}.$$

Because $67^2 < 5300 < 2 \cdot 67^2$, the only $n \leq 5300$ for which $\text{pow}(n) = 67^k$ with $k \geq 2$, is $n = 67^2$. Therefore

$$d = 79 - 4 + 2 = 77.$$

If $n = 2^j \cdot 3^k$ for $j \geq 0$ and $k \geq 1$, then $\text{pow}(n) = 3^k$. Moreover, if $0 \leq j \leq 2$ and $1 \leq k \leq 6$, or if $0 \leq j \leq 1$ and $k = 7$; then $n = 2^j \cdot 3^k \leq 2 \cdot 3^7 = 4374 < 5300$. Thus

$$b \geq 3\,(1 + 2 + \cdots + 6) + 7 + 7 = 3 \cdot 21 + 14 = 77.$$

If $n = 2^i \cdot 3^j \cdot 5^k$ for $i, j \geq 0$ and $k \geq 1$, then $\text{pow}(n) = 5^k$. Moreover, If $2^i \cdot 3^j \in \{1, 2, 3, 2^2, 2 \cdot 3, 2^3, 3^2, 2^2 \cdot 3\}$ and $1 \leq k \leq 3$, or if $2^i \cdot 3^j \in \{1, 2, 3, 2^2, 2 \cdot 3, 2^3\}$ and $k = 4$, or if $2^i \cdot 3^j = 1$ and $k = 5$; then $n = 2^i \cdot 3^j \cdot 5^k \leq 8 \cdot 5^4 = 5000 < 5300$. Thus

$$c \geq 8\,(1 + 2 + 3) + 6 \cdot 4 + 5 = 77.$$

Therefore $m = d = 77$.