UNIT 1 EXERCISES 16-20

2D GEO

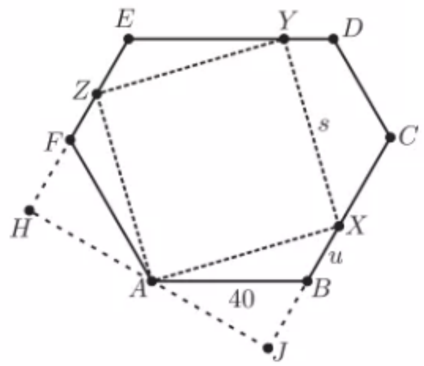
2012B 21. **Answer (A):** Extend $\overline{EF}$ to H and extend $\overline{CB}$ to J so that $\overline{HJ}$ contains A and is perpendicular to lines EF and CB. Let s be the side length of the square and let $u = BX$. Because $\angle ABJ = 60°$, it follows that $BJ = 20$ and $AJ = 20\sqrt{3}$. Then by the Pythagorean Theorem

$$AX^2 = s^2 = (20+u)^2 + (20\sqrt{3})^2.$$

Because $ABCDEF$ is equiangular, it follows that $\overline{ED} \parallel \overline{AB}$ and so $\overline{EY} \parallel \overline{AB}$. Also $\overline{ZY} \parallel \overline{AX}$ and thus it follows that $\angle EYZ = \angle BAX$ and so $\triangle EYZ \cong \triangle BAX$. Thus $EZ = u$. Also, $\angle HZA = 90° - \angle YZE = 90° - \angle AXJ = \angle JAX$; thus $\triangle AXJ \cong \triangle ZAH$ and so $ZH = 20\sqrt{3}$ and $HA = 20 + u$. Moreover, $\angle HFA = 60°$ and so $FH = \frac{HA}{\sqrt{3}} = \frac{1}{\sqrt{3}}(20+u)$. But $EZ + ZH = EF + FH$, and so

$$u + 20\sqrt{3} = 41(\sqrt{3}-1) + \frac{20+u}{\sqrt{3}}.$$

Solving for u yields $u = 21\sqrt{3} - 20$. Then $s^2 = (21\sqrt{3})^2 + (20\sqrt{3})^2 = 3 \cdot 29^2$ and therefore $s = 29\sqrt{3}$.



2014B

25. **Answer (B):** Let $O = (0, 0)$, $A = (4, 3)$, and $B = (-4, -3)$. Because $A, B \in P$ and O is the midpoint of $\overline{AB}$, it follows that $\overline{AB}$ is the latus rectum of the parabola P. Thus the directrix is parallel to $\overline{AB}$. Let T be the foot of the perpendicular from O to the directrix of P. Because $OT = OA = OB = 5$ and $\overline{OT}$ is perpendicular to $\overline{AB}$, it follows that $T = (3, -4)$. Thus the equation of the directrix is $y+4 = \frac{3}{4}(x-3)$, and in general form the equation is $4y-3x+25 = 0$.

Using the formula for the distance from a point to a line, as well as the definition of P as the locus of points equidistant from O and the directrix, the equation of P is

$$\sqrt{x^2 + y^2} = \frac{|4y - 3x + 25|}{\sqrt{4^2 + 3^2}}.$$

After squaring and rearranging, this is equivalent to

$$\begin{aligned} 25x^2 + 25y^2 &= 25(x^2 + y^2) = (4y - 3x + 25)^2 \\ &= 16y^2 + 9x^2 - 24xy + 25^2 + 50(4y - 3x), \end{aligned}$$

and

$$(4x + 3y)^2 = 25(25 + 2(4y - 3x)). \qquad (1)$$

Assume x and y are integers. Then $4x + 3y$ is divisible by 5. If $4x + 3y = 5s$ for $s \in \mathbb{Z}$, then $2s^2 = 50+16y-12x = 50+16y-3(5s-3y) = 50+25y-15s$. Thus s is divisible by 5. If $s = 5t$ for $t \in \mathbb{Z}$, then $2t^2 = 2+y-3t$, and so $y = 2t^2+3t-2$. In addition $4x = 5s - 3y = 25t - 3y = 25t - 3(2t^2 + 3t - 2) = -6t^2 + 16t + 6$, and thus t is odd. If $t = 2u + 1$ for $u \in \mathbb{Z}$, then

$$x = -6u^2 + 2u + 4 \text{ and } y = 8u^2 + 14u + 3. \qquad (2)$$

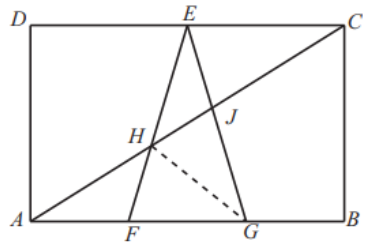
Conversely, if x and y are defined as in (2) for $u \in \mathbb{Z}$, then x and y are integers and they satisfy (1), which is the equation of P. Lastly, with $u \in \mathbb{Z}$,

$$\begin{aligned} |4x + 3y| &= \left|-24u^2 + 8u + 16 + 24u^2 + 42u + 9\right| \\ &= |50u + 25| \le 1000 \end{aligned}$$

if and only if u is an integer such that $|2u + 1| \le 39$. That is, $-20 \le u \le 19$, and so the required answer is $19 - (-21) = 40$.

2001

22. **(C)** The area of triangle EFG is $(1/6)(70) = 35/3$. Triangles AFH and CEH are similar, so $3/2 = EC/AF = EH/HF$ and $EH/EF = 3/5$. Triangles AGJ and CEJ are similar, so $3/4 = EC/AG = EJ/JG$ and $EJ/EG = 3/7$.



Since the areas of the triangles that have a common altitude are proportional to their bases, the ratio of the area of $\triangle EHJ$ to the area of $\triangle EHG$ is $3/7$, and the ratio of the area of $\triangle EHG$ to that of $\triangle EFG$ is $3/5$. Therefore, the ratio of the area of $\triangle EHJ$ to the area of $\triangle EFG$ is $(3/5)(3/7) = 9/35$. Thus, the area of $\triangle EHJ$ is $(9/35)(35/3) = 3$.

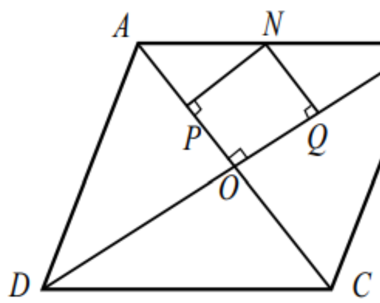
2003B

22. (C) Let O be the point of intersection of $\overline{AC}$ and $\overline{BD}$. Then AOB is a right triangle with legs $OA = 8$ and $OB = 15$. Quadrilateral $OPNQ$ is a rectangle because it has right angles at O, P, and Q. Thus $PQ = ON$, because the diagonals of a rectangle are of equal length. The minimum value of PQ is the minimum value of ON. This is achieved if and only if N is the foot of the altitude from O in triangle AOB. Writing the area of $\triangle AOB$ in two different ways yields

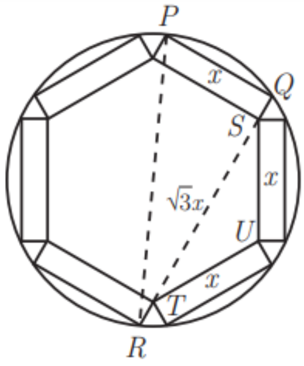
$$\frac{1}{2}AB \cdot ON = \frac{1}{2}OA \cdot OB.$$

Hence the minimum value of PQ is

$$ON = \frac{OA \cdot OB}{AB} = \frac{OA \cdot OB}{\sqrt{OA^2 + OB^2}} = \frac{8 \cdot 15}{17} = \frac{120}{17} \approx 7.06.$$



2008A



22. **Answer (C):** Select one of the mats. Let P and Q be the two corners of the mat that are on the edge of the table, and let R be the point on the edge of the table that is diametrically opposite P as shown. Then R is also a corner of a mat and $\triangle PQR$ is a right triangle with hypotenuse $PR = 8$. Let S be the inner corner of the chosen mat that lies on $\overline{QR}$, T the analogous point on the mat with corner R, and U the corner common to the other mat with corner S and the other mat with corner T. Then $\triangle STU$ is an isosceles triangle with two sides of length x and vertex angle $120°$. It follows that $ST = \sqrt{3}x$, so $QR = QS+ST+TR = \sqrt{3}x+2$. Apply the Pythagorean Theorem to $\triangle PQR$ to obtain $\left(\sqrt{3}x+2\right)^2 + x^2 = 8^2$, from which $x^2 + \sqrt{3}x - 15 = 0$. Solve for x and ignore the negative root to obtain

$$x = \frac{3\sqrt{7} - \sqrt{3}}{2}.$$

2012B 25. **Answer (B):** First note that the isosceles right triangles t can be excluded from the product because $f(t) = 1$ for these triangles. All triangles mentioned from now on are scalene right triangles. Let $O = (0, 0)$. First consider all triangles $t = \triangle ABC$ with vertices in $S \cup \{O\}$. Let R_1 be the reflection with respect to the line with equation $x = 2$. Let $A_1 = R_1(A)$, $B_1 = R_1(B)$, $C_1 = R_1(C)$, and $t_1 = \triangle A_1B_1C_1$. Note that $\triangle ABC \cong \triangle A_1B_1C_1$ with right angles at A and A_1, but the counterclockwise order of the vertices of t_1 is A_1, C_1, and B_1. Thus $f(t_1) = \tan(\angle A_1C_1B_1) = \tan(\angle ACB)$ and

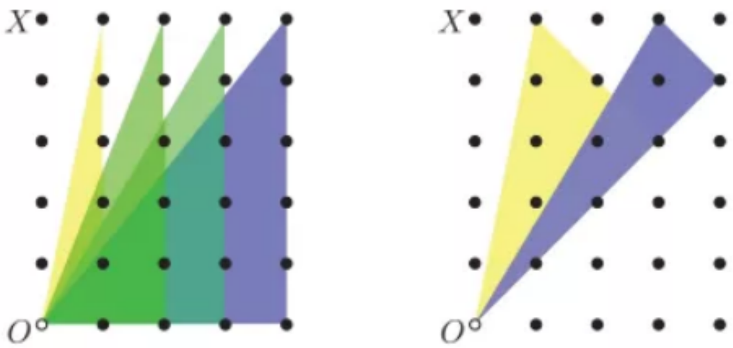
$$f(t)f(t_1) = \tan(\angle CBA)\tan(\angle ACB) = \frac{AC}{AB} \cdot \frac{AB}{AC} = 1.$$

The reflection R_1 is a bijection of $S \cup \{O\}$ and it induces a partition of the triangles in pairs (t, t_1) such that $f(t)f(t_1) = 1$. Thus the product over all triangles in $S \cup \{O\}$ is equal to 1, and thus the required product is equal to the reciprocal of $\prod_{t \in T_1} f(t)$, where T_1 is the set of triangles with vertices in $S \cup \{O\}$ having O as one vertex.

Let $S_1 = \{(x, y) : x \in \{0, 1, 2, 3, 4\}, \text{ and } y \in \{0, 1, 2, 3, 4\}\}$ and let R_2 be the reflection with respect to the line with equation $x = y$. For every right triangle $t = \triangle OBC$ with vertices B and C in S_1, let $B_2 = R_2(B)$, $C_2 = R_2(C)$, and $t_2 = \triangle OB_2C_2$. Similarly as before, R_2 is a bijection of S_1 and it induces a partition of the triangles in pairs (t, t_2) such that $f(t)f(t_2) = 1$. Thus

$\prod_{t \in T_1} f(t) = \prod_{t \in T_2} f(t)$, where T_2 is the set of triangles with vertices in $S \cup \{O\}$ with O as one vertex, and another vertex with y coordinate equal to 5.

Next, consider the reflection R_3 with respect to the line with equation $y = \frac{5}{2}$. Let $X = (0, 5)$. For every right triangle $t = \triangle OXC$ with C in S, let $C_3 = R_3(C)$, and $t_3 = \triangle OXC_3$. As before R_3 induces a partition of these triangles in pairs (t, t_3) such that $f(t)f(t_3) = 1$. Therefore to calculate $\prod_{t \in T_2} f(t)$, the only triangles left to consider are the triangles of the form $t = \triangle OYZ$ where $Y \in \{(x, 5) : x \in \{1, 2, 3, 4\}\}$ and $Z \in S \setminus \{X\}$.



The following argument shows that there are six such triangles. Because the y coordinate of Y is greater than zero, the right angle of t is not at O. The slope of the line OY has the form $\frac{5}{x}$ with $1 \le x \le 4$, so if the right angle were at Y, then the vertex Z would need to be at least 5 horizontal units away from Y, which is impossible. Therefore the right angle is at Z. There are 4 such triangles with Z on the x-axis, with vertices $O, Z = (x, 0)$, and $Y = (x, 5)$ for $1 \le x \le 4$. There are two more triangles: with vertices O, $Z = (3, 3)$, and $Y = (1, 5)$, and with vertices O, $Z = (4, 4)$, and $Y = (3, 5)$. The product of the values $f(t)$ over these six triangles is equal to

$$\frac{1}{5} \cdot \frac{2}{5} \cdot \frac{3}{5} \cdot \frac{4}{5} \cdot \frac{3\sqrt{2}}{2\sqrt{2}} \cdot \frac{4\sqrt{2}}{\sqrt{2}} = \frac{144}{625}.$$

2015A

24. **Answer (D):** There are 20 possible values for each of a and b, namely those in the set

$$S = \left\{0, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \frac{9}{5}\right\}.$$

If x and y are real numbers, then $(x+iy)^2 = x^2 - y^2 + i(2xy)$ is real if and only if $xy = 0$, that is, $x = 0$ or $y = 0$. Therefore $(x+iy)^4$ is real if and only if $x^2 - y^2 = 0$ or $xy = 0$, that is, $x = 0$, $y = 0$, or $x = \pm y$. Thus $((\cos(a\pi) + i\sin(b\pi))^4$ is a real number if and only if $\cos(a\pi) = 0$, $\sin(b\pi) = 0$, or $\cos(a\pi) = \pm\sin(b\pi)$. If $\cos(a\pi) = 0$ and $a \in S$, then $a = \frac{1}{2}$ or $a = \frac{3}{2}$ and b has no restrictions, so there are 40 pairs (a,b) that satisfy the condition. If $\sin(b\pi) = 0$ and $b \in S$, then $b = 0$ or $b = 1$ and a has no restrictions, so there are 40 pairs (a,b) that satisfy the condition, but there are 4 pairs that have been counted already, namely $(\frac{1}{2}, 0)$, $(\frac{1}{2}, 1)$, $(\frac{3}{2}, 0)$, and $(\frac{3}{2}, 1)$. Thus the total so far is $40 + 40 - 4 = 76$.

Note that $\cos(a\pi) = \sin(b\pi)$ implies that $\cos(a\pi) = \cos(\pi(\frac{1}{2} - b))$ and thus $a \equiv \frac{1}{2} - b \pmod 2$ or $a \equiv -\frac{1}{2} + b \pmod 2$. If the denominator of $b \in S$ is 3 or 5, then the denominator of a in simplified form would be 6 or 10, and so $a \notin S$. If $b = \frac{1}{2}$ or $b = \frac{3}{2}$, then there is a unique solution to either of the two congruences, namely $a = 0$ and $a = 1$, respectively. For every $b \in \{\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}\}$, there is exactly one solution $a \in S$ to each of the previous congruences. None of the solutions are equal to each other because if $\frac{1}{2} - b \equiv -\frac{1}{2} + b \pmod 2$, then $2b \equiv 1 \pmod 2$; that is, $b = \frac{1}{2}$ or $b = \frac{3}{2}$. Similarly, $\cos(a\pi) = -\sin(b\pi) = \sin(-b\pi)$ implies that $\cos(a\pi) = \cos(\pi(\frac{1}{2} + b))$ and thus $a \equiv \frac{1}{2} + b \pmod 2$ or $a \equiv -\frac{1}{2} - b \pmod 2$. If the denominator of $b \in S$ is 3 or 5, then the denominator of a would be 6 or 10, and so $a \notin S$. If $b = \frac{1}{2}$ or $b = \frac{3}{2}$, then there is a unique solution to either of the two congruences, namely $a = 1$ and $a = 0$, respectively. For every $b \in \{\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}\}$, there is exactly one solution $a \in S$ to each of the previous congruences, and, as before, none of these solutions are equal to each other. Thus there are a total of $2 + 8 + 2 + 8 = 20$ pairs $(a,b) \in S^2$ such that $\cos(a\pi) = \pm\sin(b\pi)$. The requested probability is $\frac{76+20}{400} = \frac{96}{400} = \frac{6}{25}$.

Note: By de Moivre's Theorem the fourth power of the complex number $x + iy$ is real if and only if it lies on one of the four lines $x = 0$, $y = 0$, $x = y$, or $x = -y$. Then the counting of (a,b) pairs proceeds as above.

2018B

25. **Answer (D):** Let O_i be the center of circle ω_i for $i = 1, 2, 3$, and let K be the intersection of lines O_1P_1 and O_2P_2. Because $\angle P_1P_2P_3 = 60°$, it follows that $\triangle P_2KP_1$ is a $30-60-90°$ triangle. Let $d = P_1K$; then $P_2K = 2d$ and $P_1P_2 = \sqrt{3}d$. The Law of Cosines in $\triangle O_1KO_2$ gives

$$8^2 = (d+4)^2 + (2d-4)^2 - 2(d+4)(2d-4)\cos 60°,$$

which simplifies to $3d^2 - 12d - 16 = 0$. The positive solution is $d = 2 + \frac{2}{3}\sqrt{21}$. Then $P_1P_2 = \sqrt{3}d = 2\sqrt{3} + 2\sqrt{7}$, and the required area is

$$\frac{\sqrt{3}}{4} \cdot \left(2\sqrt{3} + 2\sqrt{7}\right)^2 = 10\sqrt{3} + 6\sqrt{7} = \sqrt{300} + \sqrt{252}.$$

The requested sum is $300 + 252 = 552$.