UNIT 8 QUESTIONS 16-20

COMBINATIONS

2001

16. **(D)** Number the spider's legs from 1 through 8, and let a_k and b_k denote the sock and shoe that will go on leg k. A possible arrangement of the socks and shoes is a permutation of the sixteen symbols $a_1, b_1, \ldots a_8, b_8$, in which a_k precedes b_k for $1 \le k \le 8$. There are 16! permutations of the sixteen symbols, and a_1 precedes b_1 in exactly half of these, or $16!/2$ permutations. Similarly, a_2 precedes b_2 in exactly half of those, or $16!/2^2$ permutations. Continuing, we can conclude that a_k precedes b_k for $1 \le k \le 8$ in exactly $16!/2^8$ permutations.

2007B

16. **Answer (A):** Let r, w, and b be the number of red, white, and blue faces, respectively. Then (r, w, b) is one of 15 possible ordered triples, namely one of the three permutations of $(4, 0, 0)$, $(2, 2, 0)$, or $(2, 1, 1)$, or one of the six permutations of $(3, 1, 0)$. The number of distinguishable colorings for each of these ordered triples is the same as for any of its permutations. If $(r, w, b) = (4, 0, 0)$, then exactly one coloring is possible. If $(r, w, b) = (3, 1, 0)$, the tetrahedron can be placed with the white face down. If $(r, w, b) = (2, 2, 0)$, the tetrahedron can be placed with one white face down and the other facing forward. If $(r, w, b) = (2, 1, 1)$, the tetrahedron can be placed with the white face down and the blue face forward. Therefore there is only one coloring for each ordered triple, and the total number of distinguishable colorings is 15.

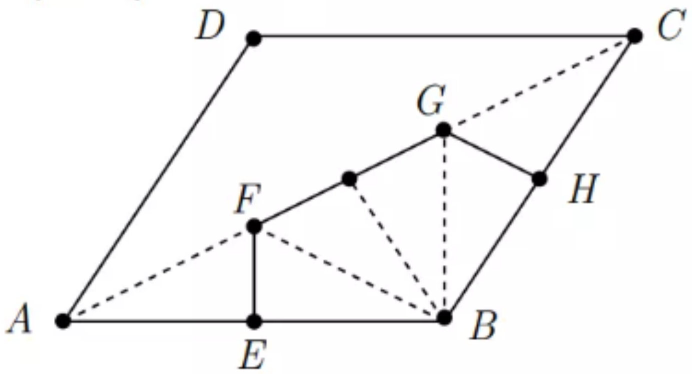
2007A

16. **Answer (C):** The set of the three digits of such a number can be arranged to form an increasing arithmetic sequence. There are 8 possible sequences with a common difference of 1, since the first term can be any of the digits 0 through 7. There are 6 possible sequences with a common difference of 2, 4 with a common difference of 3, and 2 with a common difference of 4. Hence there are 20 possible arithmetic sequences. Each of the 4 sets that contain 0 can be arranged to form $2 \cdot 2! = 4$ different numbers, and the 16 sets that do not contain 0 can be arranged to form $3! = 6$ different numbers. Thus there are a total of $4 \cdot 4 + 16 \cdot 6 = 112$ numbers with the required properties.

2011A

16. **Answer (C):** Let E and H be the midpoints of $\overline{AB}$ and $\overline{BC}$, respectively. The line drawn perpendicular to $\overline{AB}$ through E divides the rhombus into two regions: points that are closer to vertex A than B, and points that are closer to vertex B than A. Let F be the intersection of this line with diagonal $\overline{AC}$. Similarly, let point G be the intersection of the diagonal $\overline{AC}$ with the perpendicular to $\overline{BC}$ drawn from H. Then the desired region R is the pentagon $BEFGH$.

Note that $\triangle AFE$ is a $30-60-90°$ triangle with $AE = 1$. Hence the area of $\triangle AFE$ is $\frac{1}{2} \cdot 1 \cdot \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{6}$. Both $\triangle BFE$ and $\triangle BGH$ are congruent to $\triangle AFE$, so they have the same areas. Also $\angle FBG = 120° - \angle FBE - \angle GBH = 60°$, so $\triangle FBG$ is an equilateral triangle. In fact, the altitude from B to $\overline{FG}$ divides $\triangle FBG$ into two triangles, each congruent to $\triangle AFE$. Hence the area of $BEFGH$ is $4 \cdot \frac{\sqrt{3}}{6} = \frac{2\sqrt{3}}{3}$.



2012B

16. **Answer (B):** There are two cases to consider.

Case 1
Each song is liked by two of the girls. Then one of the three pairs of girls likes one of the six possible pairs of songs, one of the remaining pairs of girls likes one of the remaining two songs, and the last pair of girls likes the last song. This case can occur in $3 \cdot 6 \cdot 2 = 36$ ways.

Case 2
Three songs are each liked by a different pair of girls, and the fourth song is liked by at most one girl. There are $4! = 24$ ways to assign the songs to these four categories, and the last song can be liked by Amy, Beth, Jo, or no one. This case can occur in $24 \cdot 4 = 96$ ways.

The total number of possibilities is $96 + 36 = 132$.

2010B 17. **Answer (D):** Let a_{ij} denote the entry in row i and column j. he given conditions imply that $a_{11} = 1$, $a_{33} = 9$, and $a_{22} = 4, 5$, or 6. If $a_{22} = 4$, then $\{a_{12}, a_{21}\} = \{2, 3\}$, and the sets $\{a_{31}, a_{32}\}$ and $\{a_{13}, a_{23}\}$ are complementary subsets of $\{5, 6, 7, 8\}$. There are $\binom{4}{2} = 6$ ways to choose $\{a_{31}, a_{32}\}$ and $\{a_{13}, a_{23}\}$, and only one way to order the entries. There are 2 ways to order $\{a_{12}, a_{21}\}$, so 12 arrays with $a_{22} = 4$ meet the given conditions. Similarly, the conditions are met by 12 arrays with $a_{22} = 6$. If $a_{22} = 5$, then $\{a_{12}, a_{13}, a_{23}\}$ and $\{a_{21}, a_{31}, a_{32}\}$ are complementary subsets of $\{2, 3, 4, 6, 7, 8\}$ subject to the conditions $a_{12} < 5$, $a_{21} < 5$, $a_{32} > 5$, and $a_{23} > 5$. Thus $\{a_{12}, a_{13}, a_{23}\} \neq \{2, 3, 4\}$ or $\{6, 7, 8\}$, so its elements can be chosen in $\binom{6}{3} - 2 = 18$ ways. Both the remaining entries and the ordering of all entries are then determined, so 18 arrays with $a_{22} = 5$ meet the given conditions.

Altogether, the conditions are met by $12 + 12 + 18 = 42$ arrays.

2013A 17. **Answer (D):** For $1 \leq k \leq 11$, the number of coins remaining in the chest before the k^{th} pirate takes a share is $\frac{12}{12-k}$ times the number remaining afterward. Thus if there are n coins left for the 12^{th} pirate to take, the number of coins originally in the chest is

$$\frac{12^{11} \cdot n}{11!} = \frac{2^{22} \cdot 3^{11} \cdot n}{2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11} = \frac{2^{14} \cdot 3^7 \cdot n}{5^2 \cdot 7 \cdot 11}.$$

The smallest value of n for which this is a positive integer is $5^2 \cdot 7 \cdot 11 = 1925$. In this case there are

$$2^{14} \cdot 3^7 \cdot \frac{11!}{(12-k)! \cdot 12^{k-1}}$$

coins left for the k^{th} pirate to take, and note that this amount is an integer for each k. Hence the 12^{th} pirate receives 1925 coins.

2006B

18. **(B)** Each step changes either the x-coordinate or the y-coordinate of the object by 1. Thus if the object's final point is (a, b), then $a+b$ is even and $|a|+|b| \leq 10$. Conversely, suppose that (a, b) is a lattice point with $|a| + |b| = 2k \leq 10$. One ten-step path that ends at (a, b) begins with $|a|$ horizontal steps, to the right if $a \geq 0$ and to the left if $a < 0$. It continues with $|b|$ vertical steps, up if $b \geq 0$ and down if $b < 0$. It has then reached (a, b) in $2k$ steps, so it can finish with $5 - k$ steps up and $5 - k$ steps down. Thus the possible final points are the lattice points that have even coordinate sums and lie on or inside the square with vertices $(\pm 10, 0)$ and $(0, \pm 10)$. There are 11 such points on each of the 11 lines $x + y = 2k$, $-5 \leq k \leq 5$, for a total of 121 different points.

2010A

18. **Answer (D):** Each such path intersects the line $y = -x$ at exactly one of the points $(\pm 4, \mp 4)$, $(\pm 3, \mp 3)$, or $(\pm 2, \mp 2)$. For $j = 0$, 1, and 2, the number of paths from $(-4, 4)$ to either of $(\pm(4-j), \mp(4-j))$ is $\binom{8}{j}$, and the number of paths to $(4, 4)$ from either of $(\pm(4-j), \mp(4-j))$ is the same. Therefore the number of paths that meet the requirement is $2\left(\binom{8}{0}^2 + \binom{8}{1}^2 + \binom{8}{2}^2\right) = 2(1^2 + 8^2 + 28^2) = 1698$.

2014B 18. **Answer (B):** The circular arrangement 14352 is bad because the sum 6 cannot be achieved with consecutive numbers, and the circular arrangement 23154 is bad because the sum 7 cannot be so achieved. It remains to show that these are the only bad arrangements. Given a circular arrangement, sums 1 through 5 can be achieved with a single number, and if the sum n can be achieved, then the sum $15 - n$ can be achieved using the complementary subset. Therefore an arrangement is not bad as long as sums 6 and 7 can be achieved. Suppose 6 cannot be achieved. Then 1 and 5 cannot be adjacent, so by a suitable rotation and/or reflection, the arrangement is $1bc5e$. Furthermore, $\{b, c\}$ cannot equal $\{2, 3\}$ because $1 + 2 + 3 = 6$; similarly $\{b, c\}$ cannot equal $\{2, 4\}$. It follows that $e = 2$, which then forces the arrangement to be 14352 in order to avoid consecutive 213. This arrangement is bad. Next suppose that 7 cannot be achieved. Then 2 and 5 cannot be adjacent, so again without loss of generality the arrangement is $2bc5e$. Reasoning as before, $\{b, c\}$ cannot equal $\{3, 4\}$ or $\{1, 4\}$, so $e = 4$, and then $b = 3$ and $c = 1$, to avoid consecutive 421; therefore the arrangement is 23154, which is also bad. Thus there are only two bad arrangements up to rotation and reflection.

2003B

19. (E) Since the first term is not 1, the probability that it is 2 is $1/4$. If the first term is 2, then the second term cannot be 2. If the first term is not 2, there are four equally likely values, including 2, for the second term. Thus the probability that the second term is 2 is

$$\frac{1}{4} \cdot 0 + \frac{3}{4} \cdot \frac{1}{4} = \frac{3}{16},$$

so $a + b = 3 + 16 = 19$.

OR

The set S contains $(4)(4!) = 96$ permutations, since there are 4 choices for the first term, and for each of these choices there are $4!$ arrangements of the remaining terms. The number of permutations in S whose second term is 2 is $(3)(3!) = 18$, since there are 3 choices for the first term, and for each of these choices there are $3!$ arrangements of the last 3 terms. Thus the requested probability is $18/96 = 3/16$, and $a + b = 19$.

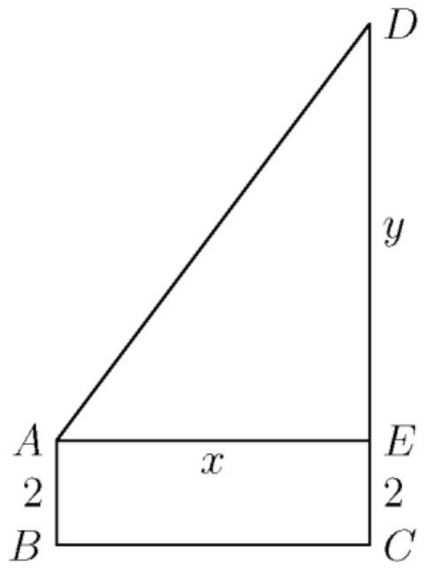
2012A

19. **Answer (B):** This situation can be modeled with a graph having these six people as vertices, in which two vertices are joined by an edge if and only if the corresponding people are internet friends. Let n be the number of friends each person has; then $1 \leq n \leq 4$. If $n = 1$, then the graph consists of three edges sharing no endpoints. There are 5 choices for Adam's friend and then 3 ways to partition the remaining 4 people into 2 pairs of friends, for a total of $5 \cdot 3 = 15$ possibilities. The case $n = 4$ is complementary, with non-friendship playing the role of friendship, so there are 15 possibilities in that case as well.

For $n = 2$, the graph must consist of cycles, and the only two choices are two triangles (3-cycles) and a hexagon (6-cycle). In the former case, there are $\binom{5}{2} = 10$ ways to choose two friends for Adam and that choice uniquely determines the triangles. In the latter case, every permutation of the six vertices determines a hexagon, but each hexagon is counted $6 \cdot 2 = 12$ times, because the hexagon can start at any vertex and be traversed in either direction. This gives $\frac{6!}{12} = 60$ hexagons, for a total of $10 + 60 = 70$ possibilities. The complementary case $n = 3$ provides 70 more. The total is therefore $15 + 15 + 70 + 70 = 170$.

2015A

19. **Answer (B):** In every such quadrilateral, $CD \geq AB$. Let E be the foot of the perpendicular from A to $\overline{CD}$; then $CE = 2$ and $AE = BC$. Let $x = AE$ and $y = DE$; then $AD = 2+y$. By the Pythagorean Theorem, $x^2 + y^2 = (2+y)^2$, or $x^2 = 4 + 4y$. Therefore x is even, say $x = 2z$, and $z^2 = 1 + y$. The perimeter of the quadrilateral is $x + 2y + 6 = 2z^2 + 2z + 4$. Increasing positive integer values of z give the required quadrilaterals, with increasing perimeter. For $z = 31$ the perimeter is 1988, and for $z = 32$ the perimeter is 2116. Therefore there are 31 such quadrilaterals.



2003A

20. (A) Since the first group of five letters contains no A's, it must contain k B's and $(5-k)$ C's for some integer k with $0 \leq k \leq 5$. Since the third group of five letters contains no C's, the remaining k C's must be in the second group, along with $(5-k)$ A's.

Similarly, the third group of five letters must contain k A's and $(5-k)$ B's. Thus each arrangement that satisfies the conditions is determined uniquely by the location of the k B's in the first group, the k C's in the second group, and the k A's in the third group.

For each k, the letters can be arranged in $\binom{5}{k}^3$ ways, so the total number of arrangements is

$$\sum_{k=0}^{5} \binom{5}{k}^3.$$

2013A

20. **Answer (B):** Consider the elements of S as integers modulo 19. Assume $a \succ b$. If $a > b$, then $a - b \leq 9$. If $a < b$, then $b - a > 9$; that is $b - a \geq 10$ and so $(a + 19) - b \leq 9$. Thus $a \succ b$ if and only if $0 < (a - b) \pmod{19} \leq 9$.

Suppose that (x, y, z) is a triple in $S \times S \times S$ such that $x \succ y$, $y \succ z$, and $z \succ x$. There are 19 possibilities for the first entry x. Once x is chosen, y can equal $x + i$ for any i, $1 \leq i \leq 9$. Then z is at most $x + 9 + i$ and at least $x + 10$, so once y is chosen, there are i possibilities for the third element z.

The number of required triples is equal to $19(1 + 2 + \cdots + 9) = 19 \cdot \frac{1}{2} \cdot 9 \cdot 10 = 19 \cdot 45 = 855$.

2016B

20. **Answer (A):** There must have been $10 + 10 + 1 = 21$ teams, and therefore there were $\binom{21}{3} = \frac{21 \cdot 20 \cdot 19}{6} = 1330$ subsets $\{A, B, C\}$ of three teams. If such a subset does not satisfy the stated condition, then it consists of a team that beat both of the others. To count such subsets, note that there are 21 choices for the winning team and $\binom{10}{2} = 45$ choices for the other two teams in the subset. This gives $21 \cdot 45 = 945$ such subsets. The required answer is $1330 - 945 = 385$. To see that such a scenario is possible, arrange the teams in a circle, and let each team beat the 10 teams that follow it in clockwise order around the circle.