UNIT 5 QUESTIONS 16-20

CO-ORD GEO

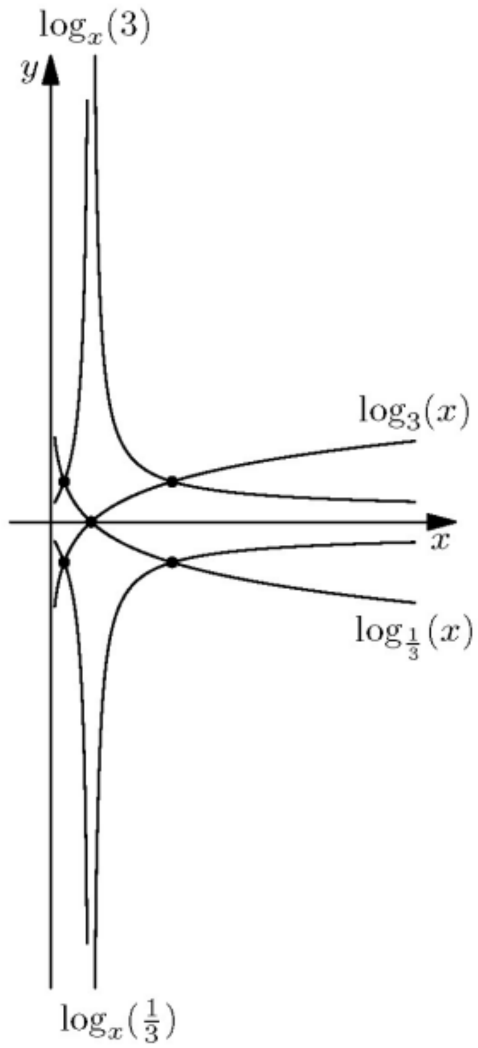
2006B 16. **(C)** Diagonals $\overline{AC}$, $\overline{CE}$, $\overline{EA}$, $\overline{AD}$, $\overline{CF}$, and $\overline{EB}$ divide the hexagon into twelve congruent $30-60-90°$ triangles, six of which make up equilateral $\triangle ACE$. Because $AC = \sqrt{7^2+1^2} = \sqrt{50}$, the area of $\triangle ACE$ is $\frac{\sqrt{3}}{4}\left(\sqrt{50}\right)^2 = \frac{25}{2}\sqrt{3}$. The area of hexagon $ABCDEF$ is $2\left(\frac{25}{2}\sqrt{3}\right) = 25\sqrt{3}$.

OR

Let O be the center of the hexagon. Then triangles ABC, CDE, and EFA are congruent to triangles AOC, COE, and EOA, respectively. Thus the area of the hexagon is twice the area of equilateral $\triangle ACE$. Then proceed as in the first solution.

2016A

16. **Answer (D):** Let $u = \log_3 x$. Then $\log_x 3 = \frac{1}{u}$, $\log_{\frac{1}{3}} x = -u$, and $\log_x \frac{1}{3} = -\frac{1}{u}$. Thus each point at which two of the graphs of the given functions intersect in the (x, y)-plane corresponds to a point at which two of the graphs of $y = u$, $y = \frac{1}{u}$, $y = -u$, and $y = -\frac{1}{u}$ intersect in the (u, y)-plane. There are 5 such points (u, y), namely $(0, 0)$, $(1, 1)$, $(-1, 1)$, $(1, -1)$, and $(-1, -1)$. The corresponding points of intersection on the graphs of the given functions are $(1, 0)$, $(3, 1)$, $(\frac{1}{3}, 1)$, $(3, -1)$, and $(\frac{1}{3}, -1)$.



2018A

16. **Answer (E):** Solving the second equation for x^2 gives $x^2 = y + a$, and substituting into the first equation gives $y^2 + y + (a - a^2) = 0$. The polynomial in y can be factored as $(y + (1 - a))(y + a)$, so the solutions are $y = a - 1$ and $y = -a$. (Alternatively, the solutions can be obtained using the quadratic formula.) The corresponding equations for x are $x^2 = 2a - 1$ and $x^2 = 0$. The second equation always has the solution $x = 0$, corresponding to the point of tangency at the vertex of the parabola $y = x^2 - a$. The first equation has 2 solutions if and only if $a > \frac{1}{2}$, corresponding to the 2 symmetric intersection points of the parabola with the circle. Thus the two curves intersect at 3 points if and only if $a > \frac{1}{2}$.

OR

Substituting the value for y from the second equation into the first equation yields

$$x^2 + \left(x^2 - a\right)^2 = a^2,$$

which is equivalent to

$$x^2\left(x^2 - (2a - 1)\right) = 0.$$

The first factor gives the solution $x = 0$, and the second factor gives 2 other solutions if $a > \frac{1}{2}$ and no other solutions if $a \le \frac{1}{2}$. Thus there are 3 solutions if and only if $a > \frac{1}{2}$.

2008B

17. **Answer (C):**

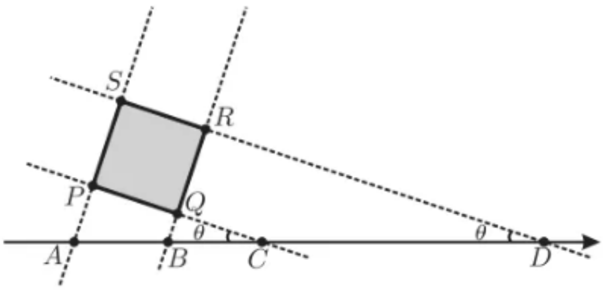
Let $A = (a, a^2)$ and $C = (c, c^2)$. Then $B = (-a, a^2)$. If either $\angle A$ or $\angle B$ is $90°$, then $c = \pm a$, but this is impossible because A, B, and C must have distinct x-coordinates. Thus $\angle C = 90°$, so $AC \perp BC$. Consequently

$$\frac{c^2 - a^2}{c - a} \cdot \frac{c^2 - a^2}{c + a} = -1,$$

from which $1 = a^2 - c^2$, which is the length of the altitude from C to $\overline{AB}$. Because $\triangle ABC$ has area 2008, it follows that $AB = 4016$, $|a| = 2008$ and $a^2 = 2008^2 = 4032064$. Therefore $c^2 = a^2 - 1 = 4032063$ and the sum of the digits of c^2 is 18.

2012B

17. **Answer (C):** Let $A=(3,0)$, $B=(5,0)$, $C=(7,0)$, $D=(13,0)$, and θ be the acute angle formed by the line PQ and the x-axis. Then $SR=PQ=AB\cos\theta=2\cos\theta$, and $SP=QR=CD\sin\theta=6\sin\theta$. Because $PQRS$ is a square, it follows that $2\cos\theta=6\sin\theta$ and $\tan\theta=\frac{1}{3}$. Therefore lines SP and RQ have slope 3, and lines SR and PQ have slope $-\frac{1}{3}$. Let the points $M=(4,0)$ and $N=(10,0)$ be the respective midpoints of segments AB and CD. Let ℓ_1 be the line through M parallel to line SP. Let ℓ_2 be the line through N parallel to line SR. Lines ℓ_1 and ℓ_2 intersect at the center of the square $PQRS$. Line ℓ_1 satisfies the equation $y=3(x-4)$, and line ℓ_2 satisfies the equation $y=-\frac{1}{3}(x-10)$. Thus the lines ℓ_1 and ℓ_2 intersect at the point $(4.6,1.8)$, and the required sum of coordinates is 6.4.



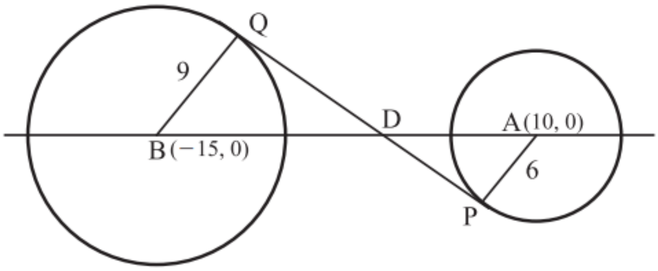
2014B

17. **Answer (E):** The line passing through point $Q=(20,14)$ with slope m has equation $y-14=m(x-20)$. The requested values for m are those for which the system

$$\begin{cases} y-14=m(x-20) \\ y=x^2 \end{cases}$$

has no solutions. Solving for y in the first equation and substituting into the second yields $m(x-20)+14=x^2$, which reduces to $x^2-mx+(20m-14)=0$. This equation has no solution for x when the discriminant is negative, that is, when $m^2-4\cdot(20m-14)=m^2-80m+56<0$. This quadratic in m is negative between its two roots $40\pm\sqrt{40^2-56}$, which are the required values of r and s. The requested sum is $r+s=2\cdot 40=80$.

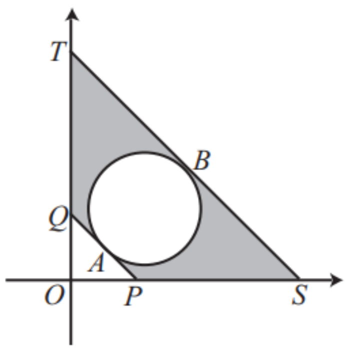
2002A 18. **(C)** The centers are at $A = (10, 0)$ and $B = (-15, 0)$, and the radii are 6 and 9, respectively. Since the internal tangent is shorter than the external tangent, $\overline{PQ}$ intersects $\overline{AB}$ at a point D that divides $\overline{AB}$ into parts proportional to the radii. The right triangles $\triangle APD$ and $\triangle BQD$ are similar with ratio of similarity $2 : 3$. Therefore, $D = (0, 0)$, $PD = 8$, and $QD = 12$. Thus $PQ = 20$.



2005B 18. **(C)** For $\triangle ABC$ to be acute, all angles must be acute. For $\angle A$ to be acute, point C must lie above the line passing through A and perpendicular to $\overline{AB}$. The segment of that line in the first quadrant lies between $P(4, 0)$ and $Q(0, 4)$. For $\angle B$ to be acute, point C must lie below the line through B and perpendicular to $\overline{AB}$. The segment of that line in the first quadrant lies between $S(14, 0)$ and $T(0, 14)$. For $\angle C$ to be acute, point C must lie outside the circle U that has $\overline{AB}$ as a diameter. Let O denote the origin. Region R, shaded below, has area equal to

$$\text{Area}(\triangle OST) - \text{Area}(\triangle OPQ) - \text{Area}(\text{Circle } U) = \frac{1}{2} \cdot 14^2 - \frac{1}{2} \cdot 4^2 - \pi \left(\frac{\sqrt{50}}{2} \right)^2$$

$$= 90 - \frac{25}{2}\pi \approx 51.$$



2016B

18. **Answer (B):** The graph of the equation is symmetric about both axes. In the first quadrant, the equation is equivalent to $x^2+y^2-x-y=0$. Completing the square gives $(x-\frac{1}{2})^2+(y-\frac{1}{2})^2=\frac{1}{2}$, so the graph in the first quadrant is an arc of the circle that is centered at $C(\frac{1}{2},\frac{1}{2})$ and contains the points $A(1,0)$ and $B(0,1)$. Because C is the midpoint of $\overline{AB}$, the arc is a semicircle. The region enclosed by the graph in the first quadrant is the union of isosceles right triangle AOB, where $O(0,0)$ is the origin, and a semicircle with diameter $\overline{AB}$. The triangle and the semicircle have areas $\frac{1}{2}$ and $\frac{1}{2}\cdot\pi\left(\frac{\sqrt{2}}{2}\right)^2=\frac{\pi}{4}$, respectively, so the area of the region enclosed by the graph in all quadrants is $4(\frac{1}{2}+\frac{\pi}{4})=\pi+2$.

2011B

19. **Answer (B):** For $0<x\le 100$, the nearest lattice point directly above the line $y=\frac{1}{2}x+2$ is $\left(x,\frac{1}{2}x+3\right)$ if x is even and $\left(x,\frac{1}{2}x+\frac{5}{2}\right)$ if x is odd. The slope of the line that contains this point and $(0,2)$ is $\frac{1}{2}+\frac{1}{x}$ if x is even and $\frac{1}{2}+\frac{1}{2x}$ if x is odd. The minimum value of the slope is $\frac{51}{100}$ if x is even and $\frac{50}{99}$ if x is odd. Therefore the line $y=mx+2$ contains no lattice point with $0<x\le 100$ for $\frac{1}{2}<m<\frac{50}{99}$.

2011

2003B 20. (B) We have

$$0 = f(-1) = -a + b - c + d \quad \text{and} \quad 0 = f(1) = a + b + c + d,$$

so $b + d = 0$. Also $d = f(0) = 2$, so $b = -2$.

OR

The polynomial is divisible by $(x+1)(x-1) = x^2 - 1$, its leading term is ax^3, and its constant term is 2, so

$$f(x) = (x^2 - 1)(ax - 2) = ax^3 - 2x^2 - ax + 2 \quad \text{and} \quad b = -2.$$

2007B 20. **Answer (D):** Two vertices of the first parallelogram are at $(0, c)$ and $(0, d)$. The x-coordinates of the other two vertices satisfy $ax + c = bx + d$ and $ax + d = bx + c$, so the x-coordinates are $\pm(c - d)/(b - a)$. Thus the parallelogram is composed of two triangles, each of which has area

$$9 = \frac{1}{2} \cdot |c - d| \cdot \left| \frac{c - d}{b - a} \right|.$$

It follows that $(c - d)^2 = 18|b - a|$. By a similar argument using the second parallelogram, $(c + d)^2 = 72|b - a|$. Subtracting the first equation from the second yields $4cd = 54|b - a|$, so $2cd = 27|b - a|$. Thus $|b - a|$ is even, and $a + b$ is minimized when $\{a, b\} = \{1, 3\}$. Also, cd is a multiple of 27, and $c + d$ is minimized when $\{c, d\} = \{3, 9\}$. Hence the smallest possible value of $a + b + c + d$ is $1 + 3 + 3 + 9 = 16$. Note that the required conditions are satisfied when $(a, b, c, d) = (1, 3, 3, 9)$.