UNIT 4 QUESTIONS 16-20

TRIANGLES

2010A

17. **Answer (E):** Triangles ABC, CDE and EFA are congruent, so $\triangle ACE$ is equilateral. Let X be the intersection of the lines AB and EF and define Y and Z similarly as shown in the figure. Because $ABCDEF$ is equiangular, it follows that $\angle XAF = \angle AFX = 60°$. Thus $\triangle XAF$ is equilateral. Let H be the midpoint of $\overline{XF}$. By the Pythagorean Theorem,

$$AE^2 = AH^2 + HE^2 = (\frac{\sqrt{3}}{2}r)^2 + (\frac{r}{2}+1)^2 = r^2 + r + 1$$

Thus, the area of $\triangle ACE$ is

$$\frac{\sqrt{3}}{4}AE^2 = \frac{\sqrt{3}}{4}(r^2 + r + 1).$$

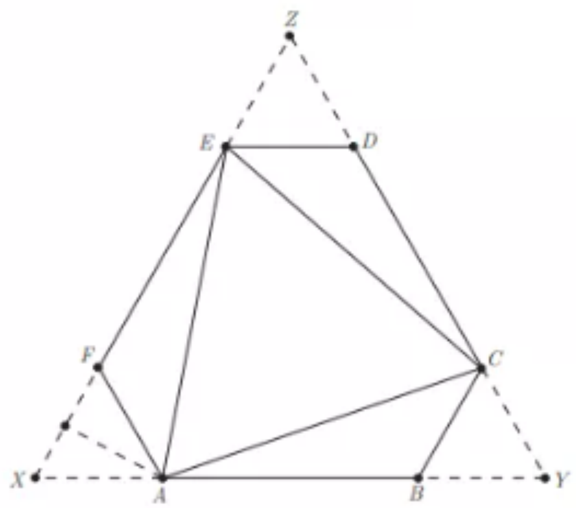
The area of hexagon $ABCDEF$ is equal to

$$[XYZ] - [XAF] - [YCB] - [ZED] = \frac{\sqrt{3}}{4}\left((2r+1)^2 - 3r^2\right) = \frac{\sqrt{3}}{4}(r^2 + 4r + 1)$$

Because $[ACE] = \frac{7}{10}[ABCDEF]$, it follows that

$$r^2 + r + 1 = \frac{7}{10}(r^2 + 4r + 1)$$

from which $r^2 - 6r + 1 = 0$ and $r = 3 \pm 2\sqrt{2}$. The sum of all possible values of r is 6.



2018A

17. **Answer (D):** Let the triangle's vertices in the coordinate plane be $(4,0)$, $(0,3)$, and $(0,0)$, with $[0,s]\times[0,s]$ representing the unplanted portion of the field. The equation of the hypotenuse is $3x+4y-12=0$, so the distance from (s,s), the corner of S closest to the hypotenuse, to this line is given by

$$\frac{|3s+4s-12|}{\sqrt{3^2+4^2}}.$$

Setting this equal to 2 and solving for s gives $s=\frac{22}{7}$ and $s=\frac{2}{7}$, and the former is rejected because the square must lie within the triangle. The unplanted area is thus $\left(\frac{2}{7}\right)^2=\frac{4}{49}$, and the requested fraction is

$$1-\frac{\frac{4}{49}}{\frac{1}{2}\cdot 4\cdot 3}=\frac{145}{147}.$$

OR

Let the given triangle be described as $\triangle ABC$ with the right angle at B and $AB=3$. Let D be the vertex of the square that is in the interior of the triangle, and let s be the edge length of the square. Then two sides of the square along with line segments $\overline{AD}$ and $\overline{CD}$ decompose $\triangle ABC$ into four regions. These regions are a triangle with base 5 and height 2, the unplanted square with side s, a right triangle with legs s and $3-s$, and a right triangle with legs s and $4-s$. The sum of the areas of these four regions is

$$\frac{1}{2}\cdot 5\cdot 2+s^2+\frac{1}{2}s(3-s)+\frac{1}{2}s(4-s)=5+\frac{7}{2}s,$$

and the area of $\triangle ABC$ is 6. Solving $5+\frac{7}{2}s=6$ for s gives $s=\frac{2}{7}$, and the solution concludes as above.

2012A

18. **Answer (A):**

Let $a = BC$, $b = AC$, and $c = AB$. Let D, E, and F be the feet of the perpendiculars from I to $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively. Because $\overline{BF}$ and $\overline{BD}$ are common tangent segments to the incircle of $\triangle ABC$, it follows that $BF = BD$. Similarly, $CD = CE$ and $AE = AF$. Thus

$$\begin{aligned} 2 \cdot BD &= BD + BF = (BC - CD) + (AB - AF) = BC + AB - (CE + AE) \\ &= a + c - b = 25 + 27 - 26 = 26, \end{aligned}$$

so $BD = 13$.

Let $s = \frac{1}{2}(a + b + c) = 39$ be the semiperimeter of $\triangle ABC$ and $r = DI$ the inradius of $\triangle ABC$. The area of $\triangle ABC$ is equal to rs and also equal to $\sqrt{s(s-a)(s-b)(s-c)}$ by Heron's formula. Thus

$$r^2 = \frac{(s-a)(s-b)(s-c)}{s} = \frac{14 \cdot 13 \cdot 12}{39} = 56.$$

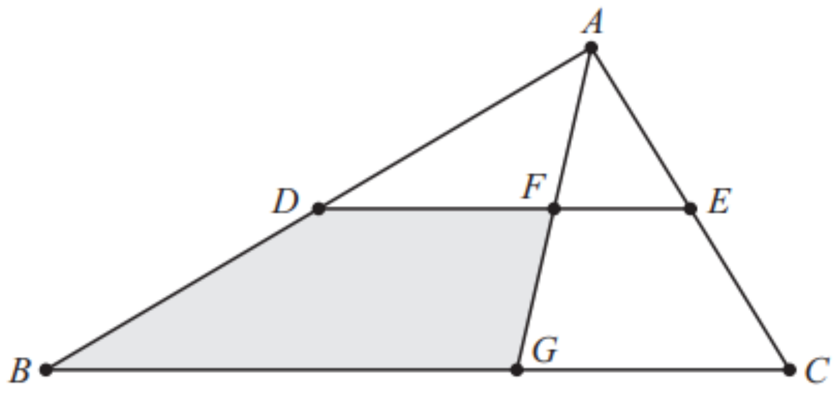
Finally, by the Pythagorean Theorem applied to the right triangle BDI, it follows that

$$BI^2 = DI^2 + BD^2 = r^2 + BD^2 = 56 + 13^2 = 56 + 169 = 225,$$

so $BI = 15$.

2018A

18. **Answer (D):** Because AB is $\frac{5}{6}$ of $AB + AC$, it follows from the Angle Bisector Theorem that DF is $\frac{5}{6}$ of DE, and BG is $\frac{5}{6}$ of BC. Because trapezoids $FDBG$ and $EDBC$ have the same height, the area of $FDBG$ is $\frac{5}{6}$ of the area of $EDBC$. Furthermore, the area of $\triangle ADE$ is $\frac{1}{4}$ of the area of $\triangle ABC$, so its area is 30, and the area of trapezoid $EDBC$ is $120 - 30 = 90$. Therefore the area of quadrilateral $FDBG$ is $\frac{5}{6} \cdot 90 = 75$.



Note: The figure (not drawn to scale) shows the situation in which $\angle ACB$ is acute. In this case $BC \approx 59.0$ and $\angle BAC \approx 151°$. It is also possible for $\angle ACB$ to be obtuse, with $BC \approx 41.5$ and $\angle BAC \approx 29°$. These values can be calculated using the Law of Cosines and the sine formula for area.

1999

19. **(C)** Let $DC = m$ and $AD = n$. By the *Pythagorean Theorem*, $AB^2 = AD^2 + DB^2$. Hence $(m+n)^2 = n^2 + 57$, which yields $m(m+2n) = 57$. Since m and n are positive integers, the only possibilities are $m = 1$, $n = 28$ and $m = 3$, $n = 8$. The second of these gives the least possible value of $AC = m + n$, namely 11.

1999

19. **(C)** Let $DC = m$ and $AD = n$. By the *Pythagorean Theorem*, $AB^2 = AD^2 + DB^2$. Hence $(m+n)^2 = n^2 + 57$, which yields $m(m+2n) = 57$. Since m and n are positive integers, the only possibilities are $m = 1$, $n = 28$ and $m = 3$, $n = 8$. The second of these gives the least possible value of $AC = m + n$, namely $\mathbf{11}$.

2007A

19. **Answer (E):** Let h be the length of the altitude from A in $\triangle ABC$. Then

$$2007 = \frac{1}{2} \cdot BC \cdot h = \frac{1}{2} \cdot 223 \cdot h,$$

so $h = 18$. Thus A is on one of the lines $y = 18$ or $y = -18$. Line DE has equation $x - y - 300 = 0$. Let A have coordinates (a, b). By the formula for the distance from a point to a line, the distance from A to line DE is $|a-b-300|/\sqrt{2}$. The area of $\triangle ADE$ is

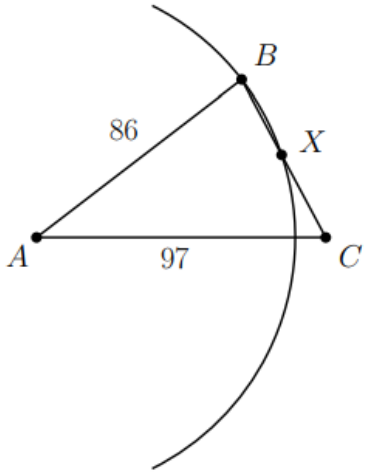
$$7002 = \frac{1}{2} \cdot \frac{|a-b-300|}{\sqrt{2}} \cdot DE = \frac{1}{2} \cdot \frac{|a \pm 18 - 300|}{\sqrt{2}} \cdot 9\sqrt{2}.$$

Thus $a = \pm 18 \pm 1556 + 300$, and the sum of the four possible values of a is $4 \cdot 300 = 1200$.

OR

As above, conclude that A is on one of the lines $y = \pm 18$. By similar reasoning, A is on one of two particular lines l_1 and l_2 parallel to $\overline{DE}$. Therefore there are four possible positions for A, determined by the intersections of the lines $y = 18$ and $y = -18$ with each of l_1 and l_2. Let the line $y = 18$ intersect l_1 and l_2 in points (x_1, y_1) and (x_2, y_2), and let the line $y = -18$ intersect l_1 and l_2 in points (x_3, y_3) and (x_4, y_4). The four points of intersection are the vertices of a parallelogram, and the center of the parallelogram has x-coordinate $(1/4)(x_1 + x_2 + x_3 + x_4)$. The center is the intersection of the line $y = 0$ and line DE. Because line DE has equation $y = x - 300$, the center of the parallelogram is $(300, 0)$. Thus the sum of all possible x-coordinates of A is $4 \cdot 300 = 1200$.

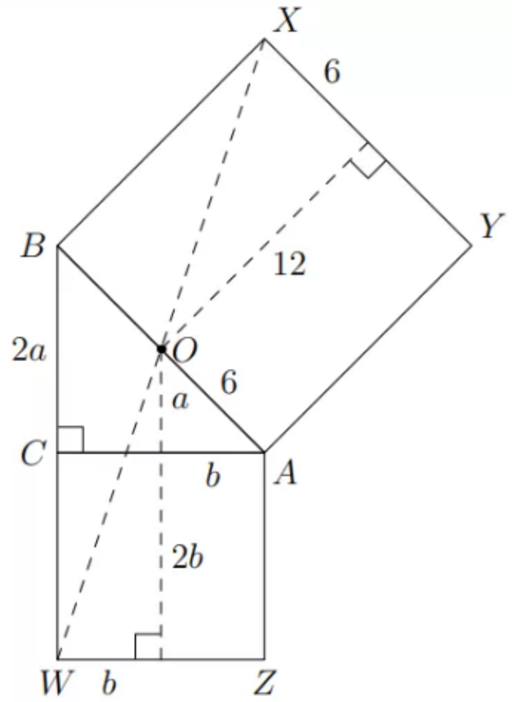
2013A 19. **Answer (D):** By the Power of a Point Theorem, $BC \cdot CX = AC^2 - r^2$ where $r = AB$ is the radius of the circle. Thus $BC \cdot CX = 97^2 - 86^2 = 2013$. Since $BC = BX + CX$ and CX are both integers, they are complementary factors of 2013. Note that $2013 = 3 \cdot 11 \cdot 61$, and $CX < BC < AB + AC = 183$. Thus the only possibility is $CX = 33$ and $BC = 61$.



2014A 19. **Answer (E):** Solve the equation for k to obtain $k = -\frac{12}{x} - 5x$. For each integer value of x except $x = 0$, there is a corresponding rational value for k. As a function of x, $|k| = \frac{12}{x} + 5x$ is increasing for $x \geq 2$. Thus by inspection, the integer values of x that ensure $|k| < 200$ satisfy the inequality $-39 \leq x \leq 39$. There are 78 such values. Assume that a and b are two different integer values of x that produce the same k. Then $k = -\frac{12}{a} - 5a = -\frac{12}{b} - 5b$, which simplifies to $(5ab - 12)(a - b) = 0$. Because $a \neq b$, it follows that $5ab = 12$, but there are no integers satisfying this equation. Thus the values of k corresponding to the 78 values of x are all distinct, and the answer is therefore 78.

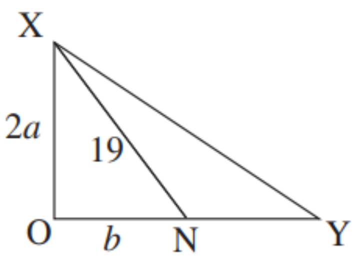
2015B

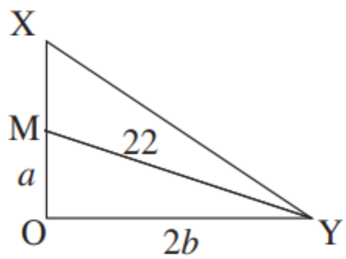
19. **Answer (C):** Let O be the center of the circle on which X, Y, Z, and W lie. Then O lies on the perpendicular bisectors of segments $\overline{XY}$ and $\overline{ZW}$, and $OX = OW$. Note that segments $\overline{XY}$ and $\overline{AB}$ have the same perpendicular bisector and segments $\overline{ZW}$ and $\overline{AC}$ have the same perpendicular bisector, from which it follows that O lies on the perpendicular bisectors of segments $\overline{AB}$ and $\overline{AC}$; that is, O is the circumcenter of $\triangle ABC$. Because $\angle C = 90°$, O is the midpoint of hypotenuse $\overline{AB}$. Let $a = \frac{1}{2}BC$ and $b = \frac{1}{2}CA$. Then $a^2 + b^2 = 6^2$ and $12^2 + 6^2 = OX^2 = OW^2 = b^2 + (a + 2b)^2$. Solving these two equations simultaneously gives $a = b = 3\sqrt{2}$. Thus the perimeter of $\triangle ABC$ is $12+2a+2b = 12 + 12\sqrt{2}$.



2002B 20. **(B)** Let $OM = a$ and $ON = b$. Then

$$19^2 = (2a)^2 + b^2 \quad \text{and} \quad 22^2 = a^2 + (2b)^2.$$





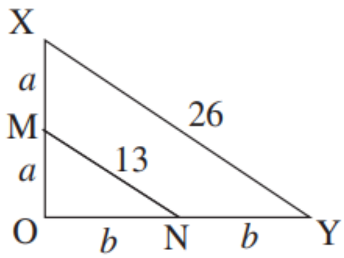
Hence

$$5(a^2 + b^2) = 19^2 + 22^2 = 845.$$

It follows that

$$MN = \sqrt{a^2 + b^2} = \sqrt{169} = 13.$$

Since $\triangle XOY$ is similar to $\triangle MON$ and $XO = 2 \cdot MO$, we have $XY = 2 \cdot MN = 26$.



2008A

20. **Answer (E):** By the Angle Bisector Theorem,

$$AD = 5 \cdot \frac{3}{3+4} = \frac{15}{7} \quad \text{and} \quad BD = 5 \cdot \frac{4}{3+4} = \frac{20}{7}.$$

To determine CD, start with the relation Area($\triangle ADC$) + Area($\triangle BCD$) = Area($\triangle ABC$) to get

$$\frac{3 \cdot CD}{2\sqrt{2}} + \frac{4 \cdot CD}{2\sqrt{2}} = \frac{3 \cdot 4}{2}.$$

This gives $CD = \frac{12\sqrt{2}}{7}$. Now use the fact that the area of a triangle is given by rs, where r is the radius of the inscribed circle and s is half the perimeter of the triangle. The ratio of the area of $\triangle ADC$ to the area of $\triangle BCD$ is the ratio of the altitudes to their common base $\overline{CD}$, which is $\frac{AD}{BD} = \frac{3}{4}$. Hence

$$\frac{3}{4} = \frac{\text{Area}(\triangle ADC)}{\text{Area}(\triangle BCD)} = \frac{r_a(3 + \frac{15}{7} + \frac{12\sqrt{2}}{7})}{r_b(4 + \frac{20}{7} + \frac{12\sqrt{2}}{7})}.$$

which yields

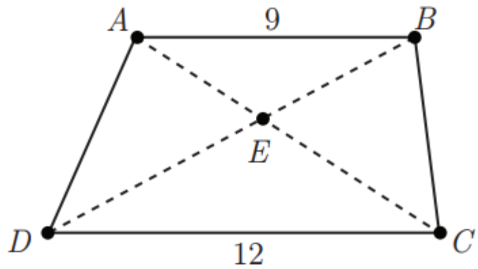
$$\frac{r_a}{r_b} = \frac{3(4+\sqrt{2})}{4(3+\sqrt{2})} = \frac{3}{28}(10-\sqrt{2}).$$

2009A

20. **Answer (E):** Because $\triangle AED$ and $\triangle BEC$ have equal areas, so do $\triangle ACD$ and $\triangle BCD$. Side $\overline{CD}$ is common to $\triangle ACD$ and $\triangle BCD$, so the altitudes from A and B to $\overline{CD}$ have the same length. Thus $\overline{AB} \parallel \overline{CD}$, so $\triangle ABE$ is similar to $\triangle CDE$ with similarity ratio

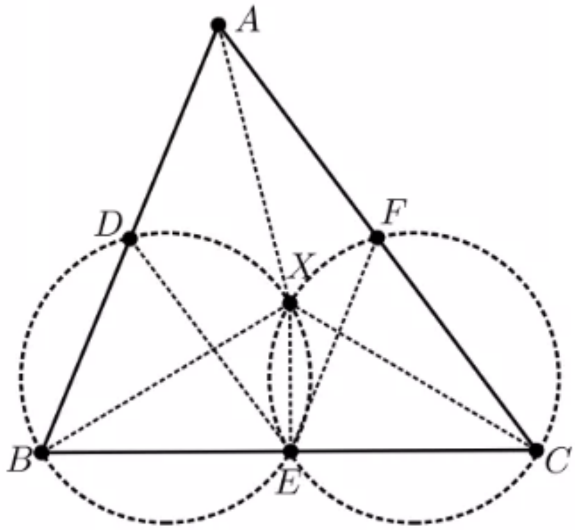
$$\frac{AE}{EC} = \frac{AB}{CD} = \frac{9}{12} = \frac{3}{4}.$$

Let $AE = 3x$ and $EC = 4x$. Then $7x = AE + EC = AC = 14$, so $x = 2$, and $AE = 3x = 6$.



2011B

20. **Answer (C):** Because $\overline{DE}$ is parallel to $\overline{AC}$ and $\overline{EF}$ is parallel to $\overline{AB}$ it follows that $\angle BDE = \angle BAC = \angle EFC$. By the Inscribed Angle Theorem, $\angle BDE = \angle BXE$ and $\angle EFC = \angle EXC$. Therefore $\angle BXE = \angle EXC$. Furthermore $BE = EC$, so by the Angle Bisector Theorem $XB = XC$. Note that $\angle BXC = 2\angle BXE = 2\angle BDE = 2\angle BAC$, and by the Inscribed Angle Theorem, it follows that X is the circumcenter of $\triangle ABC$, so $XA = XB = XC = R$ the circumradius of $\triangle ABC$.



Let $a = BC$, $b = AC$, and $c = AB$. The area of $\triangle ABC$ equals $\frac{1}{4R}(abc)$, and by Heron's Formula it also equals $\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$. Thus

$$R = \frac{abc}{4\sqrt{s\,(s-a)\,(s-b)\,(s-c)}} = \frac{13\cdot 14\cdot 15}{4\sqrt{21\cdot 8\cdot 7\cdot 6}} = \frac{65}{8},$$

and $XA + XB + XC = 3R = \frac{195}{8}$.

2015A 20. **Answer (A):** Let g and h be the lengths of the altitudes of T and T' from the sides with lengths 8 and b, respectively. The Pythagorean Theorem implies that $g = \sqrt{5^2 - 4^2} = 3$, and so the area of T is $\frac{1}{2} \cdot 8 \cdot 3 = 12$, and the perimeter is $5 + 5 + 8 = 18$. The Pythagorean Theorem implies that $h = \frac{1}{2}\sqrt{4a^2 - b^2}$. Thus $18 = 2a + b$ and

$$12 = \frac{1}{2}b \cdot \frac{1}{2}\sqrt{4a^2 - b^2} = \frac{1}{4}b\sqrt{4a^2 - b^2}.$$

Solving for a and substituting in the square of the second equation yields

$$\begin{aligned} 12^2 &= \frac{b^2}{16}\left(4a^2 - b^2\right) = \frac{b^2}{16}\left((18 - b)^2 - b^2\right) \\ &= \frac{b^2}{16} \cdot 18 \cdot (18 - 2b) = \frac{9}{4}b^2\,(9 - b)\,. \end{aligned}$$

Thus $64 - b^2(9 - b) = b^3 - 9b^2 + 64 = (b - 8)(b^2 - b - 8) = 0$. Because T and T' are not congruent, it follows that $b \neq 8$. Hence $b^2 - b - 8 = 0$ and the positive solution of this equation is $\frac{1}{2}(\sqrt{33} + 1)$. Because $25 < 33 < 36$, the solution is between $\frac{1}{2}(5 + 1) = 3$ and $\frac{1}{2}(6 + 1) = 3.5$, so the closest integer is 3.

2018A

20. **Answer (D):** It follows from the Pythagorean Theorem that $CM = MB = \frac{3}{2}\sqrt{2}$. Because quadrilateral $AIME$ is cyclic, opposite angles are supplementary and thus $\angle IME$ is a right angle. Let $x = CI$ and $y = BE$; then $AI = 3 - x$ and $AE = 3 - y$. By the Law of Cosines in $\triangle MCI$,

$$IM^2 = x^2 + \left(\frac{3}{2}\sqrt{2}\right)^2 - 2 \cdot x \cdot \frac{3}{2}\sqrt{2} \cdot \cos 45^\circ = x^2 - 3x + \frac{9}{2}.$$

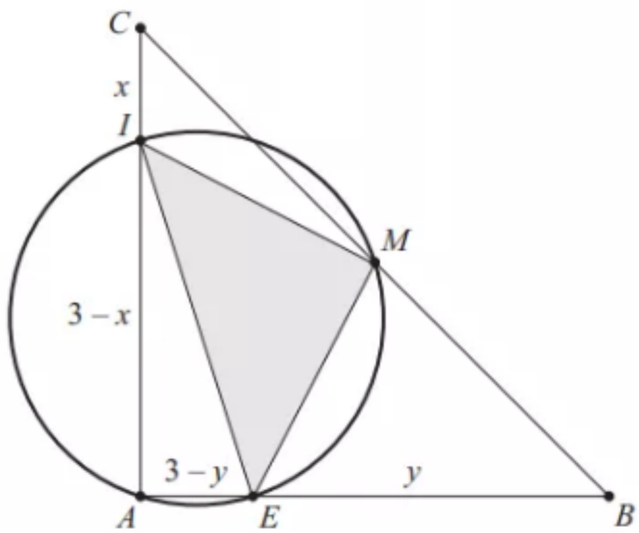
Similarly, $ME^2 = y^2 - 3y + \frac{9}{2}$. By the Pythagorean Theorem in right triangles EMI and IAE,

$$\left(x^2 - 3x + \frac{9}{2}\right) + \left(y^2 - 3y + \frac{9}{2}\right) = (3-x)^2 + (3-y)^2,$$

which simplifies to $x + y = 3$. Because the area of $\triangle EMI$ is 2, it follows that $IM^2 \cdot ME^2 = 16$. Therefore

$$\left(x^2 - 3x + \frac{9}{2}\right)\left((3-x)^2 - 3(3-x) + \frac{9}{2}\right) = 16,$$

which simplifies to $\left(x^2 - 3x + \frac{9}{2}\right)^2 = 16$. Because $y > x$, the only real solution is $x = \frac{3-\sqrt{7}}{2}$. The requested sum is $3 + 7 + 2 = 12$.

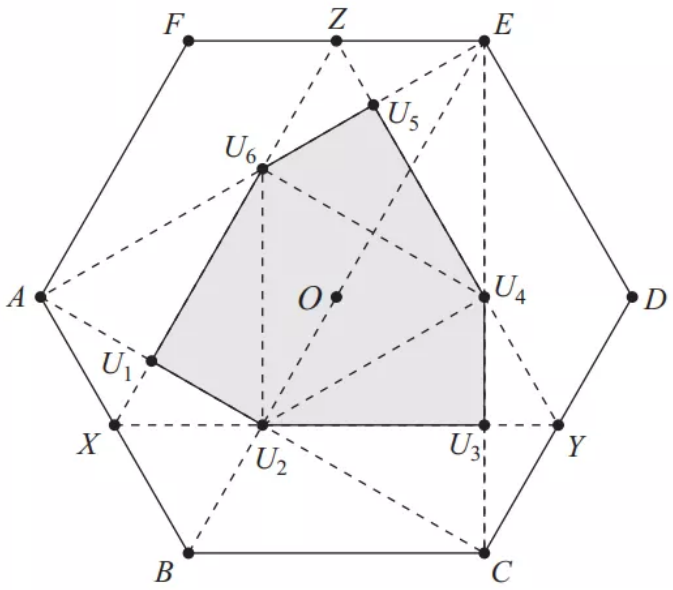


OR

Place the figure in the coordinate plane with A at $(0,0)$, B at $(3,0)$, and C at $(0,3)$. Then M is at $\left(\frac{3}{2}, \frac{3}{2}\right)$. Let $s = AE$ and $t = CI$. Then the coordinates of E are $(s, 0)$, and the coordinates of I are $(0, 3-t)$. Because $AIME$ is a cyclic quadrilateral and $\angle EAI$ is a right angle, $\angle IME$ is a right angle. Therefore $\overline{MI}$ and $\overline{ME}$ are perpendicular, so the product of their slopes is

$$\frac{\frac{3}{2}}{\quad} \cdot \frac{t - \frac{3}{2}}{\quad} = -1.$$

2018B 20. **Answer (C):** Let O be the center of the regular hexagon. Points B, O, E are collinear and $BE = BO + OE = 2$. Trapezoid $FABE$ is isosceles, and $\overline{XZ}$ is its midline. Hence $XZ = \frac{3}{2}$ and analogously $XY = ZY = \frac{3}{2}$.



Denote by U_1 the intersection of $\overline{AC}$ and $\overline{XZ}$ and by U_2 the intersection of $\overline{AC}$ and $\overline{XY}$. It is easy to see that $\triangle AXU_1$ and $\triangle U_2XU_1$ are congruent $30-60-90°$ right triangles.

By symmetry the area of the convex hexagon enclosed by the intersection of $\triangle ACE$ and $\triangle XYZ$, shaded in the figure, is equal to the area of $\triangle XYZ$ minus 3 times the area of $\triangle U_2XU_1$. The hypotenuse of $\triangle U_2XU_1$ is $XU_2 = AX = \frac{1}{2}$, so the area of $\triangle U_2XU_1$ is

$$\frac{1}{2}\cdot\frac{\sqrt{3}}{4}\cdot\left(\frac{1}{2}\right)^2 = \frac{1}{32}\sqrt{3}.$$

The area of the equilateral triangle XYZ with side length $\frac{3}{2}$ is equal to $\frac{1}{4}\sqrt{3}\cdot\left(\frac{3}{2}\right)^2 = \frac{9}{16}\sqrt{3}$. Hence the area of the shaded hexagon is

$$\frac{9}{16}\sqrt{3} - 3\cdot\frac{1}{32}\sqrt{3} = 3\sqrt{3}\left(\frac{3}{16} - \frac{1}{32}\right) = \frac{15}{32}\sqrt{3}.$$

OR

Let U_1 and U_2 be as above, and continue labeling the vertices of the shaded hexagon counterclockwise with U_3, U_4, U_5, and U_6 as shown. The area of $\triangle ACE$ is half the area of hexagon $ABCDEF$. Triangle $U_2U_4U_6$ is the midpoint triangle of $\triangle ACE$, so its area is $\frac{1}{4}$ of the area of $\triangle ACE$, and thus $\frac{1}{8}$ of the area of $ABCDEF$. Each of $\triangle U_2U_3U_4$, $\triangle U_4U_5U_6$, and $\triangle U_6U_1U_2$ is congruent to half of $\triangle U_2U_4U_6$, so the total shaded area is $\frac{5}{2}$ times the area of $\triangle U_2U_4U_6$ and therefore $\frac{5}{2}\cdot\frac{1}{8} = \frac{5}{16}$ of the area of $ABCDEF$. The area of $ABCDEF$ is $6\cdot\frac{\sqrt{3}}{4}\cdot 1^2$,