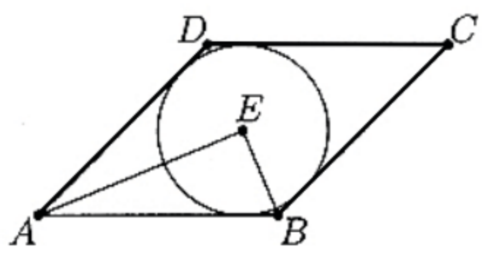
UNIT 3 QUESTIONS 16-20

2D GEO WORD

1999

16. **(C)** Let E be the intersection of the diagonals of a rhombus $ABCD$ satisfying the conditions of the problem. Because these diagonals are perpendicular and bisect each other, $\triangle ABE$ is a right triangle with sides 5, 12, and 13 and area 30. Therefore the altitude drawn to side AB is $60/13$, which is the radius of the inscribed circle centered at E.



2008B

16. **Answer (B):** Because the area of the border is half the area of the floor, the same is true of the painted rectangle. The painted rectangle measures $a - 2$ by $b - 2$ feet. Hence $ab = 2(a-2)(b-2)$, from which $0 = ab - 4a - 4b + 8$. Add 8 to each side of the equation to produce

$$8 = ab - 4a - 4b + 16 = (a-4)(b-4).$$

Because the only integer factorizations of 8 are

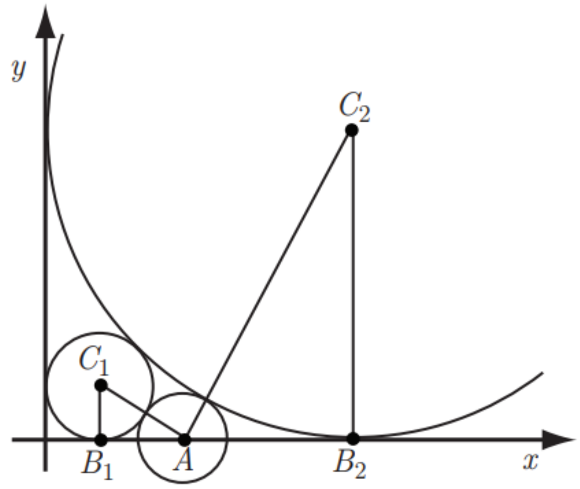
$$8 = 1 \cdot 8 = 2 \cdot 4 = (-4) \cdot (-2) = (-8) \cdot (-1),$$

and because $b > a > 0$, the only possible ordered pairs satisfying this equation for $(a-4, b-4)$ are $(1, 8)$ and $(2, 4)$. Hence (a, b) must be one of the two ordered pairs $(5, 12)$, or $(6, 8)$.

2009A 16. **Answer (D):** Let r be the radius of a circle with center C, $A = (3, 0)$, and $B = (r, 0)$. Then, $AC = 1+r$ and $CB = r$. Applying the Pythagorean Theorem to $\triangle ABC$ gives

$$AB^2 = (1 + r)^2 - r^2 = 1 + 2r.$$

Also, $AB = |3 - r|$, so $1 + 2r = (3 - r)^2$, which simplifies to $r^2 - 8r + 8 = 0$. Thus $r = 4 \pm 2\sqrt{2}$, both of which are positive, and the sum of all possible values of r is 8.



2008B

16. **Answer (B):** Because the area of the border is half the area of the floor, the same is true of the painted rectangle. The painted rectangle measures $a - 2$ by $b - 2$ feet. Hence $ab = 2(a-2)(b-2)$, from which $0 = ab - 4a - 4b + 8$. Add 8 to each side of the equation to produce

$$8 = ab - 4a - 4b + 16 = (a-4)(b-4).$$

Because the only integer factorizations of 8 are

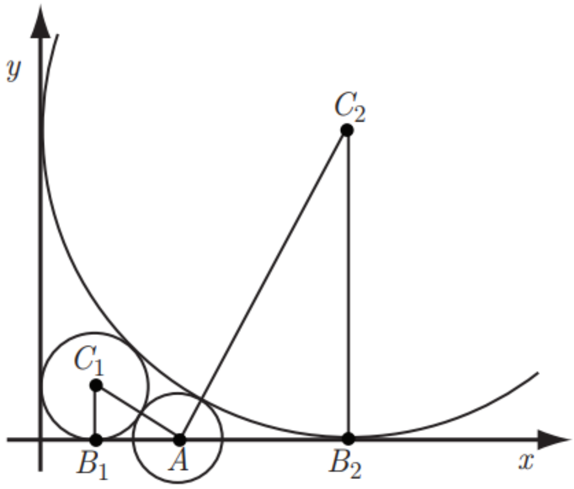
$$8 = 1 \cdot 8 = 2 \cdot 4 = (-4) \cdot (-2) = (-8) \cdot (-1),$$

and because $b > a > 0$, the only possible ordered pairs satisfying this equation for $(a-4, b-4)$ are $(1, 8)$ and $(2, 4)$. Hence (a, b) must be one of the two ordered pairs $(5, 12)$, or $(6, 8)$.

2009A 16. **Answer (D):** Let r be the radius of a circle with center C, $A = (3, 0)$, and $B = (r, 0)$. Then, $AC = 1+r$ and $CB = r$. Applying the Pythagorean Theorem to $\triangle ABC$ gives

$$AB^2 = (1+r)^2 - r^2 = 1 + 2r.$$

Also, $AB = |3 - r|$, so $1 + 2r = (3 - r)^2$, which simplifies to $r^2 - 8r + 8 = 0$. Thus $r = 4 \pm 2\sqrt{2}$, both of which are positive, and the sum of all possible values of r is 8.



2009B

16. **Answer (B):** Extend $\overline{AB}$ and $\overline{DC}$ to meet at E. Then

$$\begin{aligned}\angle BED &= 180^\circ - \angle EDB - \angle DBE \\ &= 180^\circ - 134^\circ - 23^\circ = 23^\circ.\end{aligned}$$

Thus $\triangle BDE$ is isosceles with $DE = BD$. Because $\overline{AD} \parallel \overline{BC}$, it follows that the triangles BCD and ADE are similar. Therefore

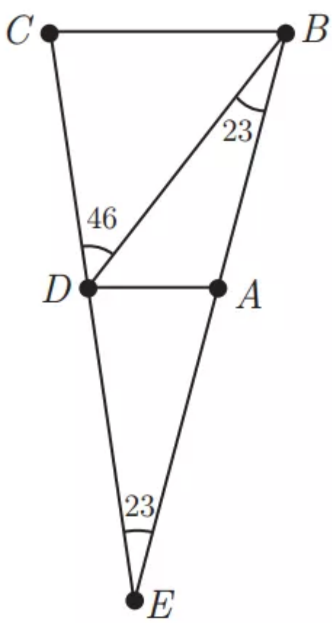
$$\frac{9}{5} = \frac{BC}{AD} = \frac{CD + DE}{DE} = \frac{CD}{BD} + 1 = CD + 1,$$

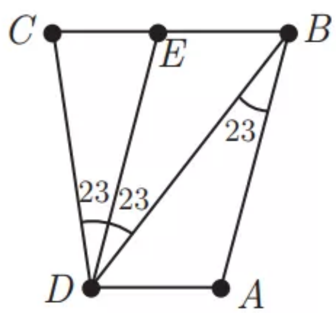
so $CD = \frac{4}{5}$.

OR

Let E be the intersection of $\overline{BC}$ and the line through D parallel to $\overline{AB}$. By construction $BE = AD$ and $\angle BDE = 23^\circ$; it follows that DE is the bisector of the angle BDC. By the Bisector Theorem we get

$$CD = \frac{CD}{BD} = \frac{EC}{BE} = \frac{BC - BE}{BE} = \frac{BC}{AD} - 1 = \frac{9}{5} - 1 = \frac{4}{5}.$$

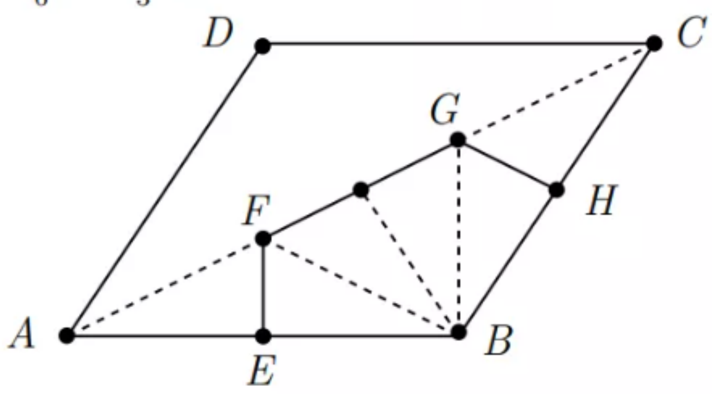




2011B

16. **Answer (C):** Let E and H be the midpoints of $\overline{AB}$ and $\overline{BC}$, respectively. The line drawn perpendicular to $\overline{AB}$ through E divides the rhombus into two regions: points that are closer to vertex A than B, and points that are closer to vertex B than A. Let F be the intersection of this line with diagonal $\overline{AC}$. Similarly, let point G be the intersection of the diagonal $\overline{AC}$ with the perpendicular to $\overline{BC}$ drawn from H. Then the desired region R is the pentagon $BEFGH$.

Note that $\triangle AFE$ is a $30-60-90°$ triangle with $AE = 1$. Hence the area of $\triangle AFE$ is $\frac{1}{2} \cdot 1 \cdot \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{6}$. Both $\triangle BFE$ and $\triangle BGH$ are congruent to $\triangle AFE$, so they have the same areas. Also $\angle FBG = 120° - \angle FBE - \angle GBH = 60°$, so $\triangle FBG$ is an equilateral triangle. In fact, the altitude from B to $\overline{FG}$ divides $\triangle FBG$ into two triangles, each congruent to $\triangle AFE$. Hence the area of $BEFGH$ is $4 \cdot \frac{\sqrt{3}}{6} = \frac{2\sqrt{3}}{3}$.



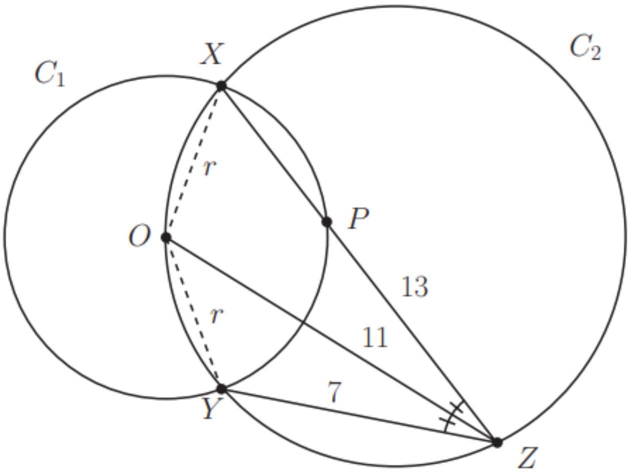
2012A 16. **Answer (E):** Let r be the radius of C_1. Because $OX = OY = r$, it follows that $\angle OZY = \angle XZO$. Applying the Law of Cosines to triangles XZO and OZY gives

$$\frac{11^2 + 13^2 - r^2}{2 \cdot 11 \cdot 13} = \cos \angle XZO = \cos \angle OZY = \frac{7^2 + 11^2 - r^2}{2 \cdot 7 \cdot 11}.$$

Solving for r^2 gives $r^2 = 30$ and so $r = \sqrt{30}$.

OR

Let P be the point on $\overline{XZ}$ such that $ZP = ZY = 7$. Because $\overline{OZ}$ is the bisector of $\angle XZY$, it follows that $\triangle OPZ \cong \triangle OYZ$. Therefore $OP = OY = r$ and thus P is on C_1. By the Power of a Point Theorem, $13 \cdot 7 = ZX \cdot ZP = OZ^2 - r^2 = 11^2 - r^2$. Solving for r^2 gives $r^2 = 30$ and so $r = \sqrt{30}$.



2014A 16. **Answer (D):** By direct multiplication, $8 \cdot 888\ldots8 = 7111\ldots104$, where the product has 2 fewer ones than the number of digits in $888\ldots8$. Because $7+4=11$, the product must have $1000-11=989$ ones, so $k-2=989$ and $k=991$.

2002B 17. **(B)** Let A be the number of square feet in Andy's lawn. Then $A/2$ and $A/3$ are the areas of Beth's lawn and Carlos' lawn, respectively, in square feet. Let R be the rate, in square feet per minute, that Carlos' lawn mower cuts. Then Beth's mower and Andy's mower cut at rates of $2R$ and $3R$ square feet per minute, respectively. Thus,

$$\text{Andy takes } \frac{A}{3R} \text{ minutes to mow his lawn,}$$

$$\text{Beth takes } \frac{A/2}{2R} = \frac{A}{4R} \text{ minutes to mow hers,}$$

and

$$\text{Carlos takes } \frac{A/3}{R} = \frac{A}{3R} \text{ minutes to mow his.}$$

Since $\frac{A}{4R} < \frac{A}{3R}$, Beth will finish first.

2011A 17. **Answer (B):** Note that

$$h_1(x) = \log_{10}\left(\frac{10^{10x}}{10}\right) = \log_{10}\left(10^{10x-1}\right) = 10x-1.$$

Therefore $h_2(x) = 10^2x-(1+10)$, $h_3(x) = 10^3x-(1+10+10^2)$, and in general,

$$h_n(x) = 10^n x - \sum_{k=0}^{n-1} 10^k.$$

Hence $h_n(1)$ is an n-digit integer whose units digit is 9 and whose other digits are all 8's. The sum of the digits of $h_{2011}(1)$ is $8 \cdot 2010 + 9 = 16{,}089$.

2016A 17. **Answer (B):** Without loss of generality, let the square and equilateral triangles have side length 6. Then the height of the equilateral triangles is $3\sqrt{3}$, and the distance of each of the triangle centers, E, F, G, and H, to the square $ABCD$ is $\sqrt{3}$. It follows that the diagonal of square $ABCD$ has length $6\sqrt{2}$, and the diagonal of square $EFGH$ has length equal to the side length of square $ABCD$ plus twice the distance from the center of an equilateral triangle to square $ABCD$ or $6 + 2\sqrt{3}$. The required ratio of the areas of the two squares is equal to the square of the ratio of the lengths of the diagonals of the two squares, or

$$\left(\frac{6+2\sqrt{3}}{6\sqrt{2}}\right)^2 = \left(\frac{3+\sqrt{3}}{3\sqrt{2}}\right)^2 = \frac{12+6\sqrt{3}}{18} = \frac{2+\sqrt{3}}{3}.$$

OR

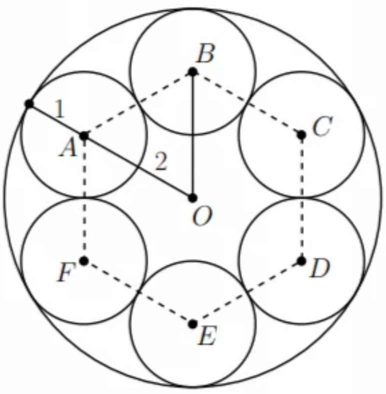
Without loss of generality, place the square in the Cartesian plane with coordinates $A(-6,0)$, $B(0,0)$, $C(0,-6)$, and $D(-6,-6)$. The center of each equilateral triangle is the point at which the medians intersect, and this point is one third of the way from the midpoint of a side of the triangle to the opposite vertex. The height of an equilateral triangle with side 6 is $3\sqrt{3}$, so the centers are $\sqrt{3}$ units from the sides of the square. Therefore the coordinates are $E(-3,\sqrt{3})$, $F(\sqrt{3},-3)$, $G(-3,-6-\sqrt{3})$, and $H(-6-\sqrt{3},-3)$. The area of square $EFGH$ is half the product of the lengths of its diagonals, or $\frac{1}{2}(6+2\sqrt{3})^2 = 24+12\sqrt{3}$. Square $ABCD$ has area 36, so the desired ratio is $\frac{2+\sqrt{3}}{3}$.

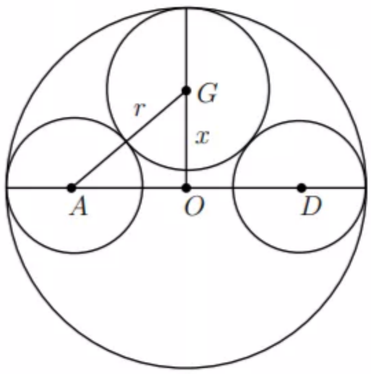
2007B 18. **Answer (C):** Let N^2 be the smaller of the two squares. Then the difference between the two squares is $(N+1)^2 - N^2 = 2N+1$. The given conditions state that

$$100a + 10b + c = N^2 + \frac{2N+1}{3} \quad \text{and} \quad 100a + 10c + b = N^2 + \frac{2(2N+1)}{3}.$$

Subtraction yields $9(c-b) = (2N+1)/3$, from which $27(c-b) = 2N+1$. If $c-b = 0$ or 2, then N is not an integer. If $c - b \geq 3$, then $N \geq 40$, so N^2 is not a three-digit integer. If $c - b = 1$, then $N = 13$. The numbers that are, respectively, one third of the way and two thirds of the way from 13^2 to 14^2 are 178 and 187, so $a + b + c = 1 + 7 + 8 = 16$.

2013A 18. **Answer (B):** Let the vertices of the regular hexagon be labeled in order A, B, C, D, E, and F. Let O be the center of the hexagon, which is also the center of the largest sphere. Let the eighth sphere have center G and radius r. Because the centers of the six small spheres are each a distance 2 from O and the small spheres have radius 1, the radius of the largest sphere is 3. Because G is equidistant from A and D, the segments $\overline{GO}$ and $\overline{AO}$ are perpendicular. Let x be the distance from G to O. Then $x + r = 3$. The Pythagorean Theorem applied to $\triangle AOG$ gives $(r+1)^2 = 2^2 + x^2 = 4 + (3-r)^2$, which simplifies to $2r + 1 = 13 - 6r$, so $r = \frac{3}{2}$. Note that this shows that the eighth sphere is tangent to $\overline{AD}$ at O.

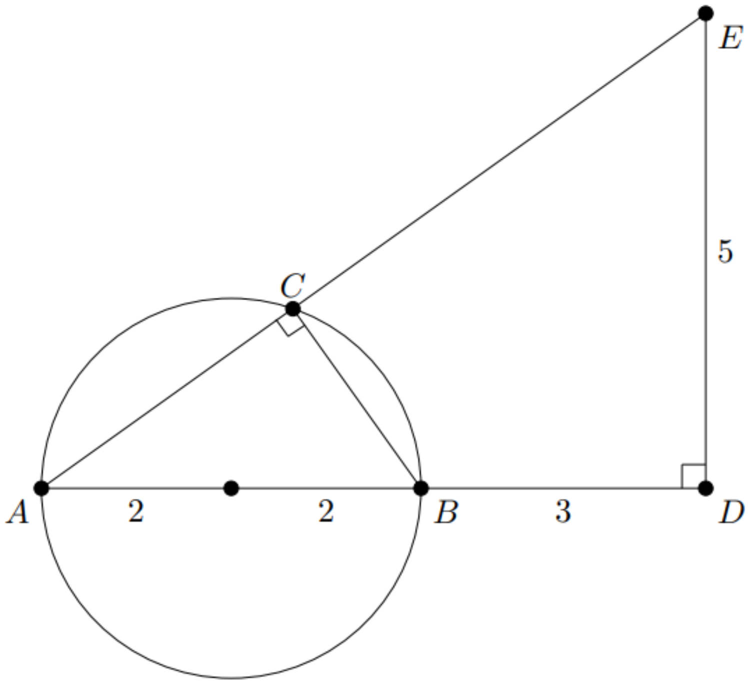




2017B 18. **Answer (D):** Because $\angle ACB$ is inscribed in a semicircle, it is a right angle. Therefore $\triangle ABC$ is similar to $\triangle AED$, so their areas are related as AB^2 is to AE^2. Because $AB^2 = 4^2 = 16$ and, by the Pythagorean Theorem,

$$AE^2 = (4+3)^2 + 5^2 = 74,$$

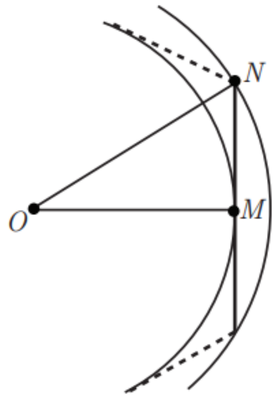
this ratio is $\frac{16}{74} = \frac{8}{37}$. The area of $\triangle AED$ is $\frac{35}{2}$, so the area of $\triangle ABC$ is $\frac{35}{2} \cdot \frac{8}{37} = \frac{140}{37}$.



2002A 19. **(D)** The equation $f(f(x)) = 6$ implies that $f(x) = -2$ or $f(x) = 1$. The horizontal line $y = -2$ intersects the graph of f twice, so $f(x) = -2$ has two solutions. Similarly, $f(x) = 1$ has 4 solutions, so there are 6 solutions of $f(f(x)) = 6$.

2009A

19. **Answer (C):** Consider a regular n-gon with side length 2. Let the radii of its inscribed and circumscribed circles be r and R, respectively. Let O be the common center of the circles, let M be the midpoint of one side of the polygon, and let N be one endpoint of that side. Then $\triangle OMN$ has a right angle at M, $MN = 1$, $OM = r$, and $ON = R$. By the Pythagorean Theorem, $R^2 - r^2 = 1$. Thus the area of the annulus between the circles is $\pi(R^2 - r^2) = \pi$ for all $n \geq 3$. Hence $A = B$.



2017A

19. **Answer (D):** In the first figure $\triangle FEB \sim \triangle DCE$, so $\frac{x}{3-x} = \frac{4-x}{x}$ and $x = \frac{12}{7}$. In the second figure, the small triangles are similar to the large one, so the lengths of the portions of the side of length 3 are as shown. Solving $\frac{3}{5}y + \frac{5}{4}y = 3$ yields $y = \frac{60}{37}$. Thus $\frac{x}{y} = \frac{12}{7} \cdot \frac{37}{60} = \frac{37}{35}$.