UNIT 22 QUESTIONS 16-20

COMPLEX

2004B

16. **(C)** From the definition of f,

$$f(x+iy)=i(x-iy)=y+ix$$

for all real numbers x and y, so the numbers that satisfy $f(z)=z$ are the numbers of the form $x+ix$. The set of all such numbers is a line through the origin in the complex plane. The set of all numbers that satisfy $|z|=5$ is a circle centered at the origin of the complex plane. The numbers satisfying both equations correspond to the points of intersection of the line and circle, of which there are two.

2018B

16. **Answer (B):** The answer would be the same if the equation were $z^8 = 81$, resulting from a horizontal translation of 6 units. The solutions to this equation are the 8 eighth roots of 81, each of which is $\sqrt[8]{3^4} = \sqrt{3}$ units from the origin. These 8 points form a regular octagon. The triangle of minimum area occurs when the vertices of the triangle are consecutive vertices of the octagon, so without loss of generality they have coordinates $A\left(\frac{1}{2}\sqrt{6}, \frac{1}{2}\sqrt{6}\right)$, $B\left(\sqrt{3}, 0\right)$, and $C\left(\frac{1}{2}\sqrt{6}, -\frac{1}{2}\sqrt{6}\right)$. This triangle has base $AC = \sqrt{6}$ and height $\sqrt{3} - \frac{1}{2}\sqrt{6}$, so its area is

$$\frac{1}{2} \cdot \sqrt{6} \cdot \left(\sqrt{3} - \frac{1}{2}\sqrt{6}\right) = \frac{3}{2}\sqrt{2} - \frac{3}{2}.$$

OR

The complex solutions form a regular octagon centered at $z = -6$. The distance from the center to any one of the vertices is $\sqrt[8]{81} =$

$\sqrt[8]{3^4} = \sqrt{3}$. By the Law of Cosines, the side length s of the octagon satisfies

$$s^2 = \left(\sqrt{3}\right)^2 + \left(\sqrt{3}\right)^2 - 2 \cdot \sqrt{3} \cdot \sqrt{3} \cdot \cos 45^\circ = 6 - 6 \cdot \frac{\sqrt{2}}{2} = 6 - 3\sqrt{2}.$$

The least possible area of $\triangle ABC$ occurs when two of the sides of $\triangle ABC$ are adjacent sides of the octagon; the angle between these two sides is 135°. The sine formula for area gives

$$\frac{1}{2} \cdot \left(6 - 3\sqrt{2}\right) \cdot \sin 135^\circ = \frac{1}{2} \cdot \left(6 - 3\sqrt{2}\right) \cdot \frac{\sqrt{2}}{2} = \frac{3}{2}\sqrt{2} - \frac{3}{2}.$$

2017A

17. **Answer (D):** The complex numbers z such that $z^{24} = 1$ are the roots of $z^{24} - 1 = (z^6 - 1)(z^6 + 1)((z^6)^2 + 1)$. The factors can have at most 6, 6, and 12, roots, respectively. Because $z^{24} - 1$ has 24 distinct roots, the factors do actually have 6, 6, and 12 distinct roots, respectively. The six roots of the first factor satisfy $z^6 = 1$, and the six roots of the second factor satisfy $z^6 = -1$. The twelve roots of the third factor satisfy $(z^6)^2 = -1$, so z^6 is never real in this case. There are $6 + 6 = 12$ roots such that z^6 is real.

OR

The complex values of z such that $z^{24} = 1$ are the 24th roots of unity. These values can be written in the form $e^{\frac{1}{12}k\pi i}$, where k is an integer between 0 and 23, inclusive. By Euler's Theorem,

$$z^6 = e^{\frac{1}{2}k\pi i} = \cos\left(\frac{1}{2}k\pi\right) + i\sin\left(\frac{1}{2}k\pi\right).$$

This quantity is a real number if and only if $\sin\left(\frac{1}{2}k\pi\right) = 0$, which occurs if and only if k is even. There are therefore 12 complex values of z such that z^6 is real.

2008B

19. **Answer (B):** Let $\alpha = a + bi$ and $\gamma = c + di$, where a, b, c, and d are real numbers. Then $f(1) = (4 + a + c) + (1 + b + d)i$, and $f(i) = (-4 - b + c) + (-1 + a + d)i$. Because both $f(1)$ and $f(i)$ are real, it follows that $a = 1 - d$ and $b = -1 - d$. Thus

$$\begin{aligned} |\alpha| + |\gamma| &= \sqrt{a^2 + b^2} + \sqrt{c^2 + d^2} \\ &= \sqrt{(1-d)^2 + (-1-d)^2} + \sqrt{c^2 + d^2} \\ &= \sqrt{2 + 2d^2} + \sqrt{c^2 + d^2}. \end{aligned}$$

The minimum value of $|\alpha| + |\gamma|$ is consequently $\sqrt{2}$, which is achieved when $c = d = 0$. In this case we also have $a = 1$ and $b = -1$.