UNIT 20 QUESTIONS 16-20

FUNCTIONS

2004A 17. **(D)** Note that

$$f(2^1) = f(2) = f(2 \cdot 1) = 1 \cdot f(1) = 2^0 \cdot 2^0 = 2^0,$$
$$f(2^2) = f(4) = f(2 \cdot 2) = 2 \cdot f(2) = 2^1 \cdot 2^0 = 2^1,$$
$$f(2^3) = f(8) = f(2 \cdot 4) = 4 \cdot f(4) = 2^2 \cdot 2^1 \cdot 2^0 = 2^{(1+2)},$$
$$f(2^4) = f(16) = f(2 \cdot 8) = 8 \cdot f(8) = 2^3 \cdot 2^2 \cdot 2^1 \cdot 2^0 = 2^{(1+2+3)},$$

and in general

$$f(2^n) = 2^{(1+2+3+\ldots+(n-1))} = 2^{n(n-1)/2}$$

It follows that $f(2^{100}) = 2^{(100)(99)/2} = 2^{4950}$.

2011B 17. **Answer (B):** Note that

$$h_1(x) = \log_{10}\left(\frac{10^{10x}}{10}\right) = \log_{10}\left(10^{10x-1}\right) = 10x - 1.$$

Therefore $h_2(x) = 10^2x-(1+10)$, $h_3(x) = 10^3x-(1+10+10^2)$, and in general,

$$h_n(x) = 10^n x - \sum_{k=0}^{n-1} 10^k.$$

Hence $h_n(1)$ is an n-digit integer whose units digit is 9 and whose other digits are all 8's. The sum of the digits of $h_{2011}(1)$ is $8 \cdot 2010 + 9 = 16{,}089$.

2006A

18. **(E)** The conditions on f imply that both

$$x = f(x) + f\left(\frac{1}{x}\right) \quad \text{and} \quad \frac{1}{x} = f\left(\frac{1}{x}\right) + f\left(\frac{1}{1/x}\right) = f\left(\frac{1}{x}\right) + f(x).$$

Thus if x is in the domain of f, then $x = 1/x$, so $x = \pm 1$.

The conditions are satisfied if and only if $f(1) = 1/2$ and $f(-1) = -1/2$.

2015B

18. **Answer (D):** To be composite, a number must have at least two prime factors, and the smallest prime number is 2. Therefore the smallest element in the range of r is $2 + 2 = 4$. To see that all integers greater than 3 are in the range, note that $r(2^n) = 2n$ for all $n \geq 2$, and $r(2^n \cdot 3) = 2n + 3$ for all $n \geq 1$.

2018B 18. **Answer (B):** Applying the recursion for several steps leads to the conjecture that

$$f(n) = \begin{cases} n+2 & \text{if } n \equiv 0 \pmod 6, \\ n & \text{if } n \equiv 1 \pmod 6, \\ n-1 & \text{if } n \equiv 2 \pmod 6, \\ n & \text{if } n \equiv 3 \pmod 6, \\ n+2 & \text{if } n \equiv 4 \pmod 6, \\ n+3 & \text{if } n \equiv 5 \pmod 6. \end{cases}$$

The conjecture can be verified using the strong form of mathematical induction with two base cases and six inductive steps. For example, if $n \equiv 2 \pmod 6$, then $n = 6k+2$ for some nonnegative integer k and

$$\begin{aligned} f(n) &= f(6k+2) \\ &= f(6k+1) - f(6k) + 6k + 2 \\ &= (6k+1) - (6k+2) + 6k + 2 \\ &= 6k+1 \\ &= n-1. \end{aligned}$$

Therefore $f(2018) = f(6 \cdot 336 + 2) = 2018 - 1 = 2017$.

OR

Note that

$$\begin{aligned} f(n) &= f(n-1) - f(n-2) + n \\ &= [f(n-2) - f(n-3) + (n-1)] - f(n-2) + n \\ &= -[f(n-4) - f(n-5) + (n-3)] + 2n - 1 \\ &= -[f(n-5) - f(n-6) + (n-4)] + f(n-5) + n + 2 \\ &= f(n-6) + 6. \end{aligned}$$

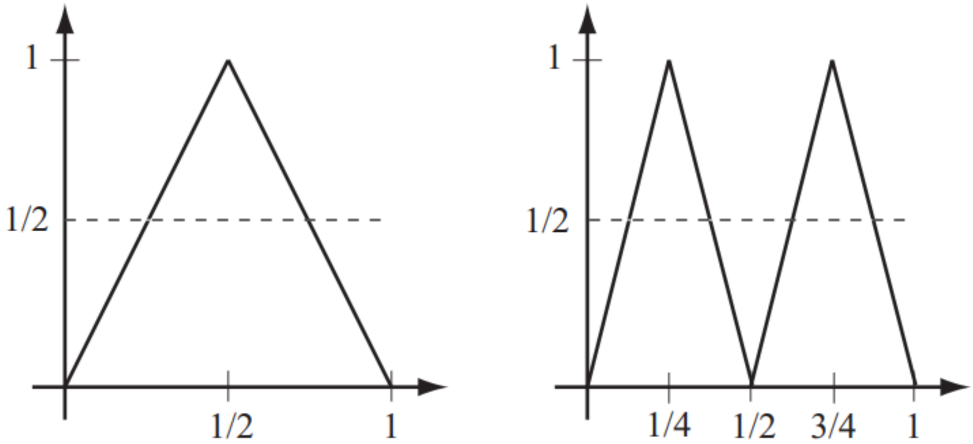
It follows that $f(2018) = f(2) + 2016 = 2017$.

2005A

20. **(E)** The graphs of $y = f(x)$ and $y = f^{[2]}(x)$ are shown below. For $n \geq 2$ we have

$$f^{[n]}(x) = f^{[n-1]}(f(x)) = \begin{cases} f^{[n-1]}(2x), & \text{if } 0 \leq x \leq \frac{1}{2}, \\ f^{[n-1]}(2-2x), & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Let $g(n)$ be the number of values of x in $[0,1]$ for which $f^{[n]}(x) = 1/2$. Then $f^{[n]}(x) = 1/2$ for $g(n-1)$ values of x in $[0,1/2]$ and $g(n-1)$ values of x in $[1/2,1]$. Furthermore $f^{[n]}(1/2) = f^{[n-1]}(1) = 0 \neq 1/2$ for $n \geq 2$. Hence $g(n) = 2g(n-1)$ for each $n \geq 2$. Because $g(1) = 2$, it follows that $g(2005) = 2^{2005}$.



2015B

20. **Answer (B):** Computing from the definition leads to the following values of $f(i,j)$ for $i = 0,1,2,3,4,5,6$ (the horizontal coordinate in the table) and $j = 0,1,2,3,4$ (the vertical coordinate).

4	0	1	1	0	3	1	1
3	4	0	4	1	1	1	1
2	3	4	2	4	3	1	1
1	2	3	0	3	1	1	1
0	1	2	3	0	3	1	1
	0	1	2	3	4	5	6

If follows that $f(i,2) = 1$ for all $i \geq 5$.

2016A 20. **Answer (A):** From the given properties, $a \diamondsuit 1 = a \diamondsuit (a \diamondsuit a) = (a \diamondsuit a) \cdot a = 1 \cdot a = a$ for all nonzero a. Then for nonzero a and b, $a = a \diamondsuit 1 = a \diamondsuit (b \diamondsuit b) = (a \diamondsuit b) \cdot b$. It follows that $a \diamondsuit b = \frac{a}{b}$. Thus

$$100 = 2016 \diamondsuit (6 \diamondsuit x) = 2016 \diamondsuit \frac{6}{x} = \frac{2016}{\frac{6}{x}} = 336x,$$

so $x = \frac{100}{336} = \frac{25}{84}$. The requested sum is $25 + 84 = 109$.