UNIT 2 QUESTIONS 16-20

3D GEO

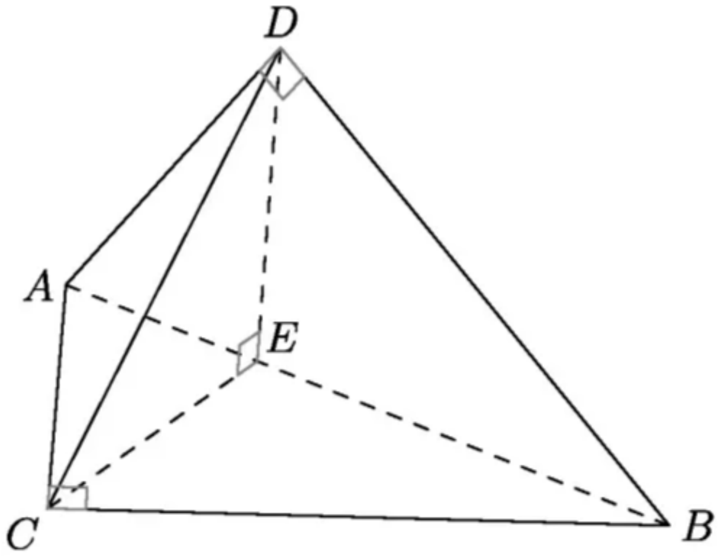
2005B

16. **(D)** The centers of the unit spheres are at the 8 points with coordinates $(\pm1, \pm1, \pm1)$, which are at a distance

$$\sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

from the origin. Hence the maximum distance from the origin to any point on the spheres is $1 + \sqrt{3}$.

2015A 16 **Answer (C):** Triangles ABC and ABD are $3-4-5$ right triangles with area 6. Let $\overline{CE}$ be the altitude of $\triangle ABC$. Then $CE = \frac{12}{5}$. Likewise in $\triangle ABD$, $DE = \frac{12}{5}$. Triangle CDE has sides $\frac{12}{5}$, $\frac{12}{5}$, and $\frac{12}{5}\sqrt{2}$, so it is an isosceles right triangle with right angle CED. Therefore $\overline{DE}$ is the altitude of the tetrahedron to base ABC. The tetrahedron's volume is $\frac{1}{3} \cdot 6 \cdot \frac{12}{5} = \frac{24}{5}$.

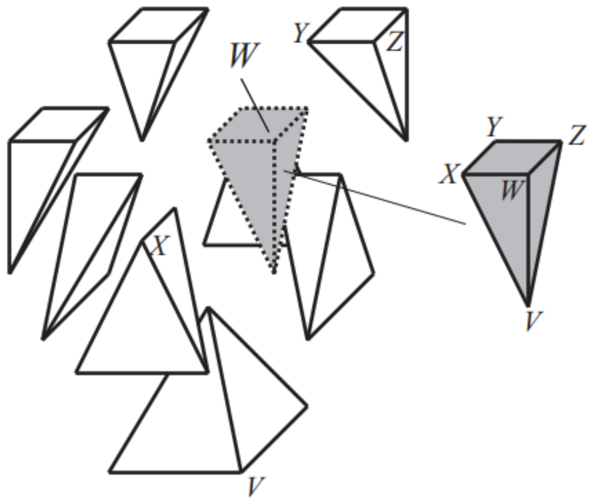


2015B 16. **Answer (C):** The distance from a vertex of the hexagon to its center is 6. The height of the pyramid can be calculated by the Pythagorean Theorem using the right triangle with other leg 6 and hypotenuse 8; it is $\sqrt{8^2 - 6^2} = 2\sqrt{7}$. The volume is then

$$\frac{1}{3}Bh = \frac{1}{3} \cdot 6 \left(6^2 \cdot \frac{\sqrt{3}}{4}\right) \cdot 2\sqrt{7} = 36\sqrt{21}.$$

2005A 17. **(A)** The piece that contains W is shown. It is a pyramid with vertices $V, W, X, Y,$ and Z. Its base $WXYZ$ is a square with sides of length $1/2$ and its altitude VW is 1. Hence the volume of this pyramid is

$$\frac{1}{3}\left(\frac{1}{2}\right)^2(1) = \frac{1}{12}.$$



2009B 17. **Answer (B):** The stripe on each face of the cube will be oriented in one of two possible directions, so there are $2^6 = 64$ possible stripe combinations on the cube. There are 3 pairs of parallel faces so, if there is an encircling stripe, then the pair of faces that do not contribute uniquely determine the stripe orientation for the remaining faces. In addition, the stripe on each face that does not contribute may be oriented in 2 different ways. Thus a total of $3\cdot 2\cdot 2 = 12$ stripe combinations on the cube result in a continuous stripe around the cube, and the requested probability is $\frac{12}{64} = \frac{3}{16}$.

OR

Without loss of generality, orient the cube so that the stripe on the top face goes from front to back. There are two mutually exclusive ways for there to be an encircling stripe: either the front, bottom, and back faces are painted to complete an encircling stripe with the top face's stripe, or the front, right, back, and left faces are painted to form an encircling stripe. The probability of the first cases is $(\frac{1}{2})^3 = \frac{1}{8}$, and the probability of the second case is $(\frac{1}{2})^4 = \frac{1}{16}$, so the answer is $\frac{1}{8} + \frac{1}{16} = \frac{3}{16}$.

OR

There are three possible orientations of an encircling stripe. For any one of these to appear, the four faces through which the stripe is to pass must be properly aligned. The probability of one such stripe alignment is $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$. Because there are 3 such possibilities, and these events are disjoint, the total probability is $3\left(\frac{1}{16}\right) = \frac{3}{16}$.

2008A

18. **Answer (C):** It may be assumed that $A = (a, 0, 0)$, $B = (0, b, 0)$, $C = (0, 0, c)$, $AB = 5$, $BC = 6$, and $CA = 7$. Then

$$a^2 + b^2 = 5^2, \quad b^2 + c^2 = 6^2, \quad \text{and} \quad a^2 + c^2 = 7^2,$$

from which

$$a^2 + b^2 + c^2 = \frac{1}{2}\left(5^2 + 6^2 + 7^2\right) = 55.$$

It follows that $a = \sqrt{55 - 6^2} = \sqrt{19}$, $b = \sqrt{55 - 7^2} = \sqrt{6}$, $c = \sqrt{55 - 5^2} = \sqrt{30}$, and the volume of tetrahedron $OABC$ can be expressed as

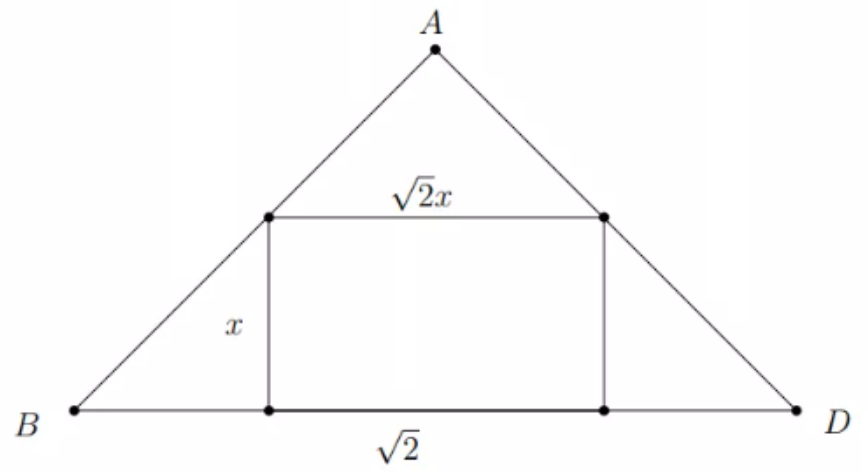
$$\frac{1}{3} \cdot OC \cdot \text{Area}(\triangle OAB) = \frac{1}{6}\sqrt{6 \cdot 19 \cdot 30} = \sqrt{95}.$$

2008B

18. **Answer (E):** Square $ABCD$ has side length 14. Let F and G be the feet of the altitudes from E in $\triangle ABE$ and $\triangle CDE$, respectively. Then $FG = 14$, $EF = 2 \cdot \frac{105}{14} = 15$ and $EG = 2 \cdot \frac{91}{14} = 13$. Because $\triangle EFG$ is perpendicular to the plane of $ABCD$, the altitude to $\overline{FG}$ is the altitude of the pyramid. By Heron's Formula, the area of $\triangle EFG$ is $\sqrt{(21)(6)(7)(8)} = 84$, so the altitude to $\overline{FG}$ is $2 \cdot \frac{84}{14} = 12$. Therefore the volume of the pyramid is $(\frac{1}{3})(196)(12) = 784$.

2011B

18. **Answer (A):** Let A be the apex of the pyramid, and let the base be the square $BCDE$. Then $AB = AD = 1$ and $BD = \sqrt{2}$, so $\triangle BAD$ is an isosceles right triangle. Let the cube have edge length x. The intersection of the cube with the plane of $\triangle BAD$ is a rectangle with height x and width $\sqrt{2}x$. It follows that $\sqrt{2} = BD = 2x + \sqrt{2}x$, from which $x = \sqrt{2} - 1$.



Hence the cube has volume

$$(\sqrt{2}-1)^3 = (\sqrt{2})^3 - 3(\sqrt{2})^2 + 3\sqrt{2} - 1 = 5\sqrt{2} - 7.$$

OR

Let A be the apex of the pyramid, let O be the center of the base, let P be the midpoint of one base edge, and let the cube intersect $\overline{AP}$ at Q. Let a coordinate plane intersect the pyramid so that O is the origin, A on the positive y-axis, and $P = \left(\frac{1}{2}, 0\right)$. Segment AP is an altitude of a lateral side of the pyramid, so $AP = \frac{\sqrt{3}}{2}$, and it follows that $A = \left(0, \frac{\sqrt{2}}{2}\right)$. Thus the equation of line AP is $y = \frac{\sqrt{2}}{2} - \sqrt{2}x$. If the side length of the cube is s, then $Q = \left(\frac{s}{2}, s\right)$, so $s = \frac{\sqrt{2}}{2} - \sqrt{2} \cdot \frac{s}{2}$. Solving gives $s = \sqrt{2} - 1$, and the result follows that in the first solution.

2004B

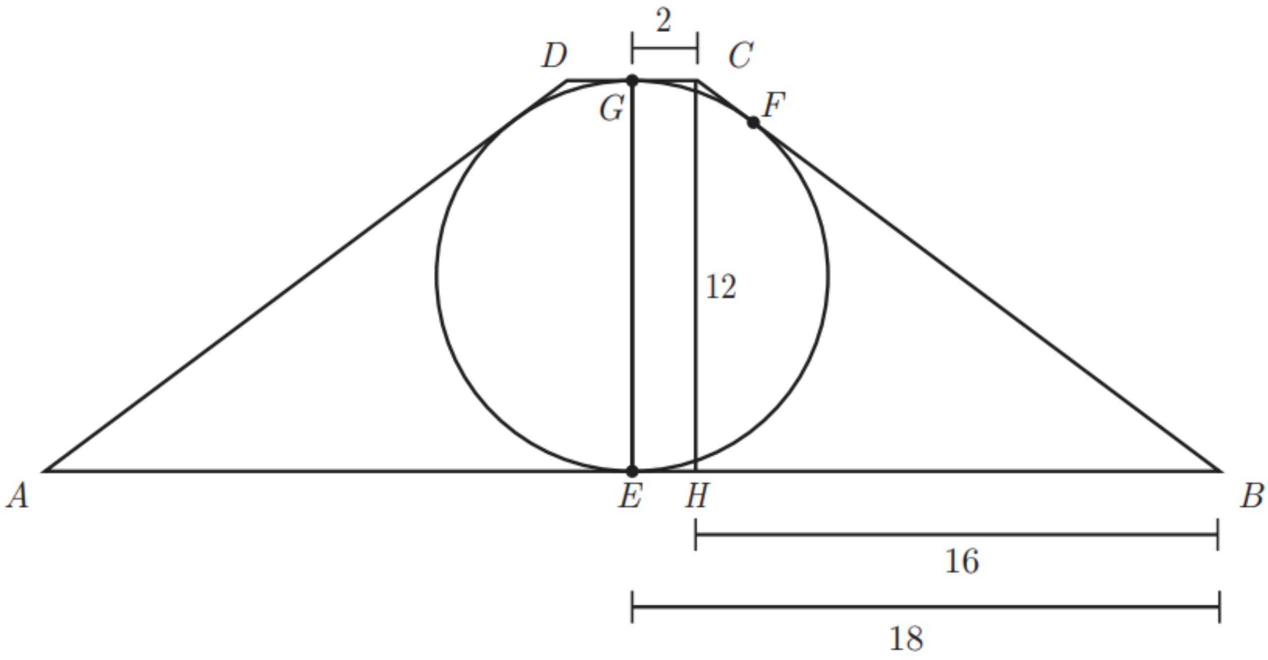
19. **(A)** Let $\overline{AB}$ and $\overline{DC}$ be parallel diameters of the bottom and top bases, respectively. A great circle of the sphere is tangent to all four sides of trapezoid $ABCD$. Let E, F, and G be the points of tangency on $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$, respectively. Then

$$FB = EB = 18 \quad \text{and} \quad FC = GC = 2,$$

so $BC = 20$. If H is on $\overline{AB}$ such that $\angle CHB$ is a right angle, then $HB = 18 - 2 = 16$. Thus

$$CH = \sqrt{20^2 - 16^2} = 12,$$

and the radius of the sphere is $(1/2)(12) = 6$.



2012B

19. **Answer (A):** Let s be the length of the octahedron's side, and let Q_i and Q'_i be the vertices of the octahedron on $\overline{P_1P_i}$ and $\overline{P'_1P'_i}$, respectively. If Q_2 and Q_3 were opposite vertices of the octahedron, then the midpoint M of $\overline{Q_2Q_3}$ would be the center of the octahedron. Because M lies on the plane $P_1P_2P_3$, the vertex of the octahedron opposite Q_4 would be outside the cube. Therefore Q_2, Q_3, and Q_4 are all adjacent vertices of the octahedron, and by symmetry so are Q'_2, Q'_3, and Q'_4. For $2 \le i < j \le 4$, the Pythagorean Theorem applied to $\triangle P_1Q_iQ_j$ gives

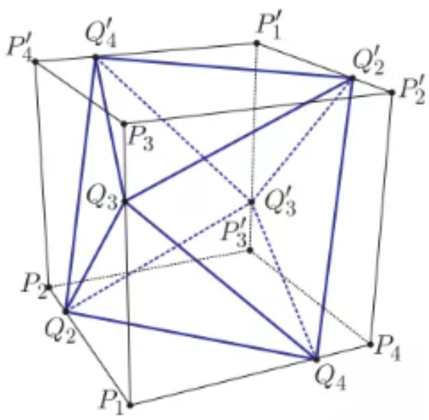
$$s^2 = (Q_iQ_j)^2 = (P_1Q_i)^2 + (P_1Q_j)^2.$$

It follows that $P_1Q_2 = P_1Q_3 = P_1Q_4 = \frac{\sqrt{2}}{2}s$, and by symmetry, $P'_1Q'_2 = P'_1Q'_3 = P'_1Q'_4 = \frac{\sqrt{2}}{2}s$. Consequently Q_i and Q'_i are opposite vertices of the octahedron. The Pythagorean Theorem on $\triangle Q_2P_2P'_3$ and $\triangle Q'_3P'_3Q_2$ gives

$$(Q_2P'_3)^2 = (P_2P'_3)^2 + (Q_2P_2)^2 = 1 + \left(1 - \frac{\sqrt{2}}{2}s\right)^2 \quad \text{and}$$

$$s^2 = (Q_2Q'_3)^2 = (P'_3Q'_3)^2 + (Q_2P'_3)^2 = \left(1 - \frac{\sqrt{2}}{2}s\right)^2 + 1 + \left(1 - \frac{\sqrt{2}}{2}s\right)^2.$$

Solving for s gives $s = \frac{3\sqrt{2}}{4}$.

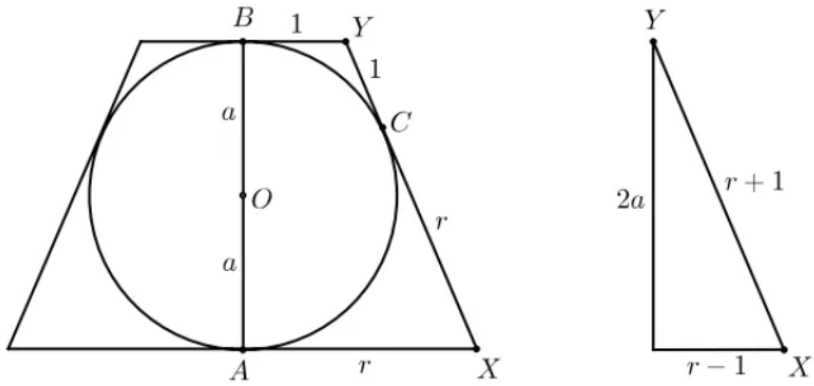


2014B

19. **Answer (E):** Assume without loss of generality that the radius of the top base of the truncated cone (frustum) is 1. Denote the radius of the bottom base by r and the radius of the sphere by a. The figure on the left is a side view of the frustum. Applying the Pythagorean Theorem to the triangle on the right yields $r = a^2$. The volume of the frustum is

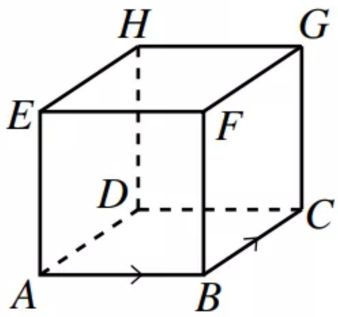
$$\frac{1}{3}\pi(r^2 + r \cdot 1 + 1^2) \cdot 2a = \frac{1}{3}\pi(a^4 + a^2 + 1) \cdot 2a.$$

Setting this equal to twice the volume of the sphere, $\frac{4}{3}\pi a^3$, and simplifying gives $a^4 - 3a^2 + 1 = 0$, or $r^2 - 3r + 1 = 0$. Therefore $r = \frac{3+\sqrt{5}}{2}$.



2006A

20. **(C)** At each vertex there are three possible locations that the bug can travel to in the next move, so the probability that the bug will visit three different vertices after two moves is $2/3$. Label the first three vertices that the bug visits as $A, B,$ and C, in that order. In order to visit every vertex, the bug must travel from C to either G or D.



The bug travels to G with probability $1/3$, and from there the bug must visit the vertices F, E, H, and D in that order. Each of these choices has probability $1/3$ of occurring. So the probability that the path continues in the form

$$C \to G \to F \to E \to H \to D$$

is $\left(\frac{1}{3}\right)^5$.

Alternatively, the bug can travel from C to D and then from D to H. Each of these occurs with probability $1/3$. From H the bug could go either to G or to E, with probability $2/3$, and from there to the two remaining vertices, each with probability $1/3$. So the probability that the path continues in one of the forms

$$C \to D \to H \begin{array}{l} \nearrow E \to F \to G \\ \searrow G \to F \to E \end{array}$$

is $\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^4$.

Hence the bug will visit every vertex in seven moves with probability

$$\left(\frac{2}{3}\right)\left[\left(\frac{1}{3}\right)^5 + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^4\right] = \left(\frac{2}{3}\right)\left(\frac{1}{3} + \frac{2}{3}\right)\left(\frac{1}{3}\right)^4 = \frac{2}{243}.$$

OR

From a given starting point there are 3^7 possible walks of seven moves for the bug, all of them equally likely. If such a walk visits every vertex exactly once, there are three choices for the first move and, excluding a return to the start, two choices for the second. Label the first three vertices visited as $A, B,$ and C, in that order, and label the other vertices as shown. The bug must go to either G or D on its third move. In the first case it must then visit vertices $F, E, H,$ and D in order. In the second case it must visit either $H, E, F,$ and G or $H, G, F,$ and E in order. Thus there are $3 \cdot 2 \cdot 3 = 18$ walks that visit every vertex exactly once, so the required probability is $18/3^7 = 2/243$.

2007A 20. **Answer (B):** Removing the corners removes two segments of equal length from each edge of the cube. Call that length x. Then each octagon has side length $\sqrt{2}x$, and the cube has edge length $1 = (2+\sqrt{2})\,x$, so

$$x = \frac{1}{2+\sqrt{2}} = \frac{2-\sqrt{2}}{2}.$$

Each removed corner is a tetrahedron whose altitude is x and whose base is an isosceles right triangle with leg length x. Thus the total volume of the eight tetrahedra is

$$8 \cdot \frac{1}{3} \cdot x \cdot \frac{1}{2}x^2 = \frac{1}{6}\left(2-\sqrt{2}\right)^3 = \frac{10-7\sqrt{2}}{3}.$$

2009B 20. **Answer (C):** Each edge of Q is cut by two planes, so R has 200 vertices. Three edges of R meet at each vertex, so R has $\frac{1}{2} \cdot 3 \cdot 200 = 300$ edges.

OR

At each vertex, as many new edges are created by this process as there are original edges meeting that vertex. Thus the total number of new edges is the total number of endpoints of the original edges, which is 200. A middle portion of each original edge is also present in R, so R has $100 + 200 = 300$ edges.

OR

Euler's Polyhedron Formula applied to Q gives $n - 100 + F = 2$, where F is the number of faces of Q. Each edge of Q is cut by two planes, so R has 200 vertices. Each cut by a plane P_k creates an additional face on R, so Euler's Polyhedron Formula applied to R gives $200 - E + (F + n) = 2$, where E is the number of edges of R. Subtracting the first equation from the second gives $300 - E = 0$, so $E = 300$.