UNIT 19 QUESTIONS 16-20

QUADRATICS

2014B

16. **Answer (E):** Because $P(0) = k$, it follows that $P(x) = ax^3 + bx^2 + cx + k$. Thus $P(1) = a+b+c+k = 2k$ and $P(-1) = -a+b-c+k = 3k$. Adding these equations gives $2b = 3k$. Hence

$$\begin{aligned} P(2) + P(-2) &= (8a + 4b + 2c + k) + (-8a + 4b - 2c + k) \\ &= 8b + 2k = 12k + 2k = 14k. \end{aligned}$$

OR

Let $(P(-2), P(-1), P(0), P(1), P(2)) = (r, 3k, k, 2k, s)$. The sequence of first differences of consecutive values is $(3k-r, -2k, k, s-2k)$, the sequence of second differences is $(r-5k, 3k, s-3k)$, and the sequence of third differences is $(8k-r, s-6k)$. Because P is a cubic polynomial, the third differences are equal, so $P(-2) + P(2) = r + s = 14k$.

1999

17. **(C)** From the hypothesis, $P(19) = 99$ and $P(99) = 19$. Let

$$P(x) = (x-19)(x-99)Q(x) + ax + b,$$

where a and b are constants and $Q(x)$ is a polynomial. Then

$$99 = P(19) = 19a + b \text{ and } 19 = P(99) = 99a + b.$$

It follows that $99a - 19a = 19 - 99$, hence $a = -1$ and $b = 99 + 19 = 118$. Thus the remainder is $-x + 118$.

2004B

2004B 18. **(E)** Let $B = (a, b)$ and $A = (-a, -b)$. Then

$$4a^2 + 7a - 1 = b \quad \text{and} \quad 4a^2 - 7a - 1 = -b.$$

Subtracting gives $b = 7a$, so $4a^2 + 7a - 1 = 7a$. Thus

$$a^2 = \frac{1}{4} \quad \text{and} \quad b^2 = (7a)^2 = \frac{49}{4},$$

so

$$AB = 2\sqrt{a^2 + b^2} = 2\sqrt{\frac{50}{4}} = 5\sqrt{2}.$$

2007A 18. **Answer (D):** Because $f(x)$ has real coefficients and $2i$ and $2+i$ are zeros, so are their conjugates $-2i$ and $2-i$. Therefore

$$\begin{aligned} f(x) = (x+2i)(x-2i)(x-(2+i))(x-(2-i)) &= (x^2+4)(x^2-4x+5) \\ &= x^4 - 4x^3 + 9x^2 - 16x + 20. \end{aligned}$$

Hence $a + b + c + d = -4 + 9 - 16 + 20 = 9$.

OR

As in the first solution,

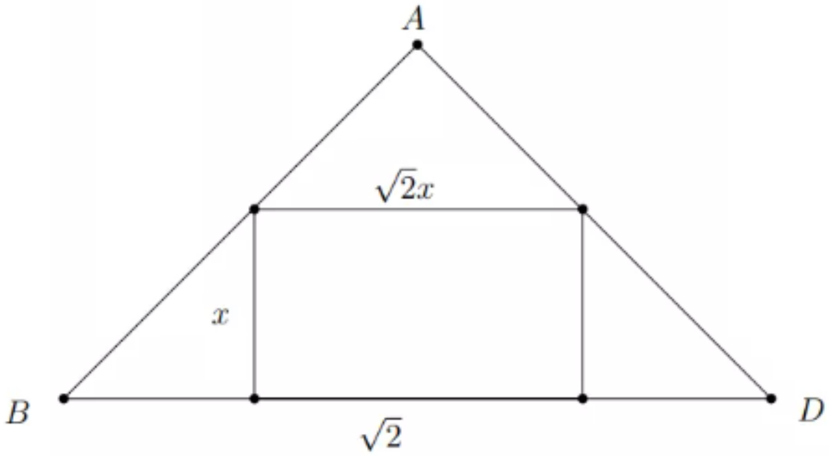
$$f(x) = (x+2i)(x-2i)(x-(2+i))(x-(2-i)),$$

so

$$a+b+c+d = f(1)-1 = (1+2i)(1-2i)(-1-i)(-1+i)-1 = (1+4)(1+1)-1 = 9.$$

2011A

18. **Answer (A):** Let A be the apex of the pyramid, and let the base be the square $BCDE$. Then $AB = AD = 1$ and $BD = \sqrt{2}$, so $\triangle BAD$ is an isosceles right triangle. Let the cube have edge length x. The intersection of the cube with the plane of $\triangle BAD$ is a rectangle with height x and width $\sqrt{2}x$. It follows that $\sqrt{2} = BD = 2x + \sqrt{2}x$, from which $x = \sqrt{2} - 1$.



Hence the cube has volume

$$(\sqrt{2}-1)^3 = (\sqrt{2})^3 - 3(\sqrt{2})^2 + 3\sqrt{2} - 1 = 5\sqrt{2} - 7.$$

OR

Let A be the apex of the pyramid, let O be the center of the base, let P be the midpoint of one base edge, and let the cube intersect $\overline{AP}$ at Q. Let a coordinate plane intersect the pyramid so that O is the origin, A on the positive y-axis, and $P = \left(\frac{1}{2}, 0\right)$. Segment AP is an altitude of a lateral side of the pyramid, so $AP = \frac{\sqrt{3}}{2}$, and it follows that $A = \left(0, \frac{\sqrt{2}}{2}\right)$. Thus the equation of line AP is $y = \frac{\sqrt{2}}{2} - \sqrt{2}x$. If the side length of the cube is s, then $Q = \left(\frac{s}{2}, s\right)$, so $s = \frac{\sqrt{2}}{2} - \sqrt{2} \cdot \frac{s}{2}$. Solving gives $s = \sqrt{2} - 1$, and the result follows that in the first solution.

2015A 18. **Answer (C):** The zeros of f are integers and their sum is a, so a is an integer. If r is an integer zero, then $r^2 - ar + 2a = 0$ or

$$a = \frac{r^2}{r-2} = r + 2 + \frac{4}{r-2}.$$

So $\frac{4}{r-2} = a - r - 2$ must be an integer, and the possible values of r are 6, 4, 3, 1, 0, and -2. The possible values of a are 9, 8, 0, and -1, all of which yield integer zeros of f, and their sum is 16.

OR

As above, a must be an integer. The function f has zeros at

$$x = \frac{a \pm \sqrt{a^2 - 8a}}{2}.$$

These values are integers only if $a^2 - 8a = w^2$ for some integer w. Solving for a in terms of w gives $a = 4 \pm \sqrt{16 + w^2}$, so $16 + w^2$ must be a perfect square. The only integer solutions for w are 0 and ± 3, from which it follows that the values of a are 0, 8, 9, and -1, all of which yield integer values of x. The requested sum is 16.

2003A 19. (D) The original parabola has equation $y = a(x-h)^2 + k$, for some a, h, and k, with $a \neq 0$. The reflected parabola has equation $y = -a(x-h)^2 - k$. The translated parabolas have equations

$$f(x) = a(x - h \pm 5)^2 + k \quad \text{and} \quad g(x) = -a(x - h \mp 5)^2 - k,$$

so

$$(f+g)(x) = \pm 20a(x-h).$$

Since $a \neq 0$, the graph is a non-horizontal line.

2001 19. **(A)** The sum and product of the zeros of $P(x)$ are $-a$ and $-c$, respectively. Therefore,

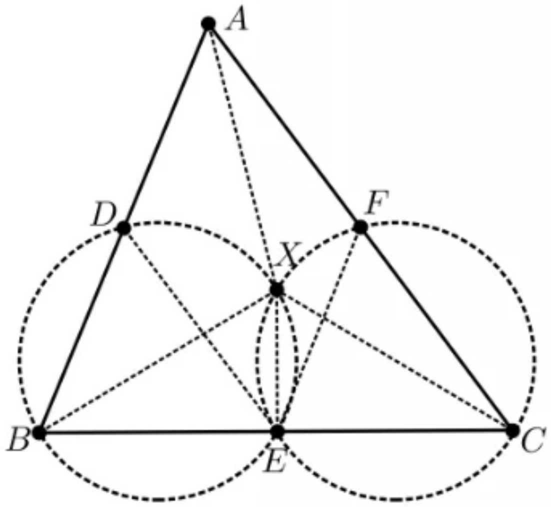
$$-\frac{a}{3} = -c = 1 + a + b + c.$$

Since $c = P(0)$ is the y-intercept of $y = P(x)$, it follows that $c = 2$. Thus $a = 6$ and $b = -11$.

2009B 19. **Answer (E):** Note that $f(n) = n^4 + 40n^2 + 400 - 400n^2 = (n^2+20)^2 - (20n)^2 = (n^2 + 20n + 20)(n^2 - 20n + 20)$. Because the first factor is greater than 1, the product cannot be prime unless the second factor is 1. The solutions of the equation $n^2 - 20n + 20 = 1$ are 1 and 19. The values of $f(1) = 1^2 + 20 \cdot 1 + 20 = 41$ and $f(19) = 19^2 + 20 \cdot 19 + 20 = 761$ are prime, and the requested sum is $41 + 761 = 802$.

2011A

20. **Answer (C):** Because $\overline{DE}$ is parallel to $\overline{AC}$ and $\overline{EF}$ is parallel to $\overline{AB}$ it follows that $\angle BDE = \angle BAC = \angle EFC$. By the Inscribed Angle Theorem, $\angle BDE = \angle BXE$ and $\angle EFC = \angle EXC$. Therefore $\angle BXE = \angle EXC$. Furthermore $BE = EC$, so by the Angle Bisector Theorem $XB = XC$. Note that $\angle BXC = 2\angle BXE = 2\angle BDE = 2\angle BAC$, and by the Inscribed Angle Theorem, it follows that X is the circumcenter of $\triangle ABC$, so $XA = XB = XC = R$ the circumradius of $\triangle ABC$.



Let $a = BC$, $b = AC$, and $c = AB$. The area of $\triangle ABC$ equals $\frac{1}{4R}(abc)$, and by Heron's Formula it also equals $\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$. Thus

$$R = \frac{abc}{4\sqrt{s\,(s-a)\,(s-b)\,(s-c)}} = \frac{13 \cdot 14 \cdot 15}{4\sqrt{21 \cdot 8 \cdot 7 \cdot 6}} = \frac{65}{8},$$

and $XA + XB + XC = 3R = \frac{195}{8}$.

2012A

20. **Answer (B):**

A factor in the product defining $P(x)$ has degree 2012 if and only if the sum of the exponents in x is equal to 2012. Because there is only one way to write 2012 as a sum of distinct powers of 2, namely the one corresponding to its binary expansion $2012 = 11111011100_2$, it follows that the coefficient of x^{2012} is equal to $2^0 \cdot 2^1 \cdot 2^5 = 2^6$.

Note: In general, if $0 \le n \le 2047$ and $n = \sum_{j\in A} 2^j$ for $A \subseteq \{0,1,2,\ldots,10\}$, then the coefficient of x^n is equal to 2^a where $a = \binom{11}{2} - \sum_{j\in A} j$.