UNIT 18 QUESTIONS 16-20

SEQUENCE AND SERIES

1999

2008A

16. **Answer (D):** The first three terms of the sequence can be written as $3\log a + 7\log b$, $5\log a + 12\log b$, and $8\log a + 15\log b$. The difference between consecutive terms can be written either as

$$(5\log a + 12\log b) - (3\log a + 7\log b) = 2\log a + 5\log b$$

or as

$$(8\log a + 15\log b) - (5\log a + 12\log b) = 3\log a + 3\log b.$$

Thus $\log a = 2\log b$, so the first term of the sequence is $13\log b$, and the difference between consecutive terms is $9\log b$. Hence the 12^{th} term is

$$(13 + (12-1)\cdot 9)\log b = 112\log b = \log(b^{112}).$$

2016B

2008A

17. **Answer (D):** If a_1 is even, then $a_2 = (a_1/2) < a_1$, so the required condition is not met. If $a_1 \equiv 1 \pmod 4$, then $a_2 = 3a_1 + 1$ is a multiple of 4, so $a_3 = (3a_1+1)/2$, and $a_4 = (3a_1+1)/4 \le a_1$. Hence the required condition is also not met in this case. If $a_1 \equiv 3 \pmod 4$, then a_2 is even but not a multiple of 4. It follows that $a_3 = (3a_1+1)/2 > a_1$, and a_3 is odd, so $a_4 = 3a_3 + 1 > a_3 > a_1$. Because 2008 is a multiple of 4, a total of $\frac{2008}{4} = 502$ possible values of a_1 are congruent to $3 \pmod 4$. These 502 values of a_1 meet the required condition.

Note: It is a famous unsolved problem to show whether or not the number 1 must be a term of this sequence for every choice of a_1 .

2009A 17. **Answer (C):** The sum of the first series is

$$\frac{a}{1-r_1} = r_1,$$

from which $r_1^2 - r_1 + a = 0$, and $r_1 = \frac{1}{2}(1 \pm \sqrt{1-4a})$. Similarly, $r_2 = \frac{1}{2}(1 \pm \sqrt{1-4a})$. Because r_1 and r_2 must be different, $r_1 + r_2 = 1$. Such series exist as long as $0 < a < \frac{1}{4}$.

1999 20. **(E)** For $n \geq 3$,

$$a_n = \frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}.$$

Thus $(n-1)a_n = a_1 + a_2 + \cdots + a_{n-1}$. It follows that

$$a_{n+1} = \frac{a_1 + a_2 + \cdots + a_{n-1} + a_n}{n} = \frac{(n-1) \cdot a_n + a_n}{n} = a_n,$$

for $n \geq 3$. Since $a_9 = 99$ and $a_1 = 19$, it follows that

$$99 = a_3 = \frac{19 + a_2}{2},$$

and hence that $a_2 = 179$. (The sequ is $19, 179, 99, 99, \ldots$.)

2010A 20. **Answer (C):** Because $a_n = 1+(n-1)d_1$ and $b_n = 1+(n-1)d_2$ for some integers d_1 and d_2, it follows that $n-1$ is a factor of $\gcd(a_n - 1, b_n - 1)$. The ordered pair (a_n, b_n) must be one of $(2, 1005)$, $(3, 670)$, $(5, 402)$, $(6, 335)$, $(10, 201)$, $(15, 134)$, or $(30, 67)$. For every pair except the sixth pair, the numbers $a_n - 1$ and $b_n - 1$ are relatively prime, so $n = 2$. In the exceptional case, $\gcd(15-1, 134-1) = 7$. The sequences defined by $a_n = 2n - 1$ and $b_n = 19n - 18$ satisfy the conditions, so $n = 8$.

2010B