UNIT 16 QUESTIONS 16-20

ALGEBRA

2002A

17. **(B)** First, observe that 4, 6, and 8 cannot be the units digit of any two-digit prime, so they must contribute at least $40 + 60 + 80 = 180$ to the sum. The remaining digits must contribute at least $1 + 2 + 3 + 5 + 7 + 9 = 27$ to the sum. Thus, the sum must be at least 207, and we can achieve this minimum only if we can construct a set of three one-digit primes and three two-digit primes. Using the facts that nine is not prime and neither two nor five can be the units digit of any two-digit prime, we can construct the sets $\{2, 3, 5, 41, 67, 89\}$, $\{2, 3, 5, 47, 61, 89\}$, or $\{2, 5, 7, 43, 61, 89\}$, each of which yields a sum of 207.

2005B 17. **(B)** The given equation is equivalent to

$$\log_{10}\left(2^a \cdot 3^b \cdot 5^c \cdot 7^d\right) = 2005, \quad \text{so} \quad 2^a \cdot 3^b \cdot 5^c \cdot 7^d = 10^{2005} = 2^{2005} \cdot 5^{2005}.$$

Let M be the least common denominator of a, b, c and d. It follows that

$$2^{Ma} \cdot 3^{Mb} \cdot 5^{Mc} \cdot 7^{Md} = 2^{2005M} \cdot 5^{2005M}.$$

Since the exponents are all integers, the Fundamental Theorem of Arithmetic implies that

$$Ma = 2005M, \quad Mb = 0, \quad Mc = 2005M, \quad \text{and} \quad Md = 0.$$

Hence the only solution is $(a, b, c, d) = (2005, 0, 2005, 0)$.

1999 18. **(E)** Note that the range of $\log x$ on the interval $(0, 1)$ is the set of all negative numbers, infinitely many of which are zeros of the cosine function. In fact, since $\cos(x) = 0$ for all x of the form $\frac{\pi}{2} \pm n\pi$,

$$\begin{aligned} f(10^{\frac{\pi}{2}-n\pi}) &= \cos(\log(10^{\frac{\pi}{2}-n\pi})) \\ &= \cos\left(\frac{\pi}{2} - n\pi\right) \\ &= 0 \end{aligned}$$

for all positive integers n.

2005A 18. **(A)** Of the numbers less than 1000, 499 of them are divisible by two, 333 are divisible by 3, and 199 are divisible by 5. There are 166 multiples of 6, 99 multiples of 10, and 66 multiples of 15. And there are 33 numbers that are divisible by 30. So by the Inclusion-Exclusion Principle there are

$$499 + 333 + 199 - 166 - 99 - 66 + 33 = 733$$

numbers that are divisible by at least one of 2, 3, or 5. Of the remaining $999 - 733 = 266$ numbers, 165 are primes other than 2, 3, or 5. Note that 1 is neither prime nor composite. This leaves exactly 100 prime-looking numbers.

2009A

18. **Answer (B):** Note that $I_k = 2^{k+2} \cdot 5^{k+2} + 2^6$. For $k < 4$, the first term is not divisible by 2^6, so $N(k) < 6$. For $k > 4$, the first term is divisible by 2^7, but the second term is not, so $N(k) < 7$. For $k = 4$, $I_4 = 2^6(5^6 + 1)$, and because the second factor is even, $N(4) \geq 7$. In fact the second factor is a sum of cubes so

$$(5^6 + 1) = ((5^2)^3 + 1^3) = (5^2 + 1)((5^2)^2 - 5^2 + 1).$$

The factor $5^2 + 1 = 26$ is divisible by 2 but not 4, and the second factor is odd, so $5^6 + 1$ contributes one more factor of 2. Hence the maximum value for $N(k)$ is 7.

2012B

18. **Answer (B):** If $a_1 = 1$, then the list must be an increasing sequence. Otherwise let $k = a_1$. Then the numbers 1 through $k - 1$ must appear in increasing order from right to left, and the numbers from k through 10 must appear in increasing order from left to right. For $2 \leq k \leq 10$ there are $\binom{9}{k-1}$ ways to choose positions in the list for the numbers from 1 through $k - 1$, and the positions of the remaining numbers are then determined. The number of lists is therefore

$$1 + \sum_{k=2}^{10} \binom{9}{k-1} = \sum_{k=0}^{9} \binom{9}{k} = 2^9 = 512.$$

2005A

19. **(B)** Because the odometer uses only 9 digits, it records mileage in base-9 numerals, except that its digits 5, 6, 7, 8, and 9 represent the base-9 digits 4, 5, 6, 7, and 8. Therefore the mileage is

$$2004_{\text{base } 9} = 2 \cdot 9^3 + 4 = 2 \cdot 729 + 4 = 1462.$$

OR

The number of miles traveled is the same as the number of integers between 1 and 2005, inclusive, that do not contain the digit 4. First consider the integers less than 2000. There are two choices for the first digit, including 0, and 9 choices for each of the other three. Because one combination of choices is 0000, there are $2 \cdot 9^3 - 1 = 1457$ positive integers less than 2000 that do not contain the digit 4. There are 5 integers between 2000 and 2005, inclusive, that do not have a 4 as a digit, so the car traveled $1457 + 5 = 1462$ miles.

2017B 19. **Answer (C):** The remainder when N is divided by 5 is clearly 4. A positive integer is divisible by 9 if and only if the sum of its digits is divisible by 9. The sum of the digits of N is $4(0+1+2+\cdots+9)+10\cdot1+10\cdot2+10\cdot3+(4+0)+(4+1)+(4+2)+(4+3)+(4+4)=270$, so N must be a multiple of 9. Then $N-9$ must also be a multiple of 9, and the last digit of $N-9$ is 5, so it is also a multiple of 5. Thus $N-9$ is a multiple of 45, and N leaves a remainder of 9 when divided by 45.

2012B 20. **Answer (D):** Let $ABCD$ be a trapezoid with $\overline{AB}\parallel\overline{CD}$ and $AB<CD$. Let E be the point on $\overline{CD}$ such that $CE=AB$. Then $ABCE$ is a parallelogram. Set $AB=a$, $BC=b$, $CD=c$, and $DA=d$. Then the side lengths of $\triangle ADE$ are b, d, and $c-a$. If one of b or d is equal to 11, say $b=11$ by symmetry, then $d+(c-a)\le 7+(5-3)<11=d$, which contradicts the triangle inequality. Thus $c=11$. There are three cases to consider, namely, $a=3$, $a=5$, and $a=7$.

If $a=3$, then $\triangle ADE$ has side lengths 5, 7, and 8 and by Heron's formula its area is

$$\frac{1}{4}\sqrt{(5+7+8)(7+8-5)(8+5-7)(5+7-8)}=10\sqrt{3}.$$

The area of $\triangle AEC$ is $\frac{3}{8}$ of the area of $\triangle ADE$, and triangles ABC and AEC have the same area. It follows that the area of the trapezoid is $\frac{1}{2}(35\sqrt{3})$.

If $a=5$, then $\triangle ADE$ has side lengths 3, 6, and 7, and area

$$\frac{1}{4}\sqrt{(3+6+7)(6+7-3)(7+3-6)(3+6-7)}=4\sqrt{5}.$$

The area of $\triangle AEC$ is $\frac{5}{6}$ of the area of $\triangle ADE$, and triangles ABC and AEC have the same area. It follows that the area of the trapezoid is $\frac{1}{3}(32\sqrt{5})$.

If $a=7$, then $\triangle ADE$ has side lengths 3, 4, and 5. Hence this is a right trapezoid with height 3 and base lengths 7 and 11. This trapezoid has area $\frac{1}{2}(3(7+11))=27$.

The sum of the three possible areas is $\frac{35}{2}\sqrt{3}+\frac{32}{3}\sqrt{5}+27$. Hence $r_1=\frac{35}{2}$, $r_2=\frac{32}{3}$, $r_3=27$, $n_1=3$, $n_2=5$, and $r_1+r_2+r_3+n_1+n_2=\frac{35}{2}+\frac{32}{3}+27+3+5=63+\frac{1}{6}$. Thus the required integer is 63.