UNIT 13 QUESTIONS 16-20

ARITHMETIC

2018B

17. **Answer (A):** The first inequality is equivalent to $9p > 5q$, and because both sides are integers, it follows that $9p - 5q \geq 1$. Similarly, $4q - 7p \geq 1$. Now

$$\begin{aligned}\frac{1}{63} = \frac{4}{7} - \frac{5}{9} &= \left(\frac{p}{q} - \frac{5}{9}\right) + \left(\frac{4}{7} - \frac{p}{q}\right) \\ &= \frac{9p - 5q}{9q} + \frac{4q - 7p}{7q} \\ &\geq \frac{1}{9q} + \frac{1}{7q} \\ &= \frac{16}{63q}.\end{aligned}$$

Thus $q \geq 16$. Because

$$\frac{8}{16} < \frac{5}{9} < \frac{9}{16} < \frac{4}{7} < \frac{10}{16},$$

the fraction $\frac{9}{16}$ lies in the required interval, but $\frac{8}{16}$ and $\frac{10}{16}$ do not. Therefore when q is as small as possible, $q = 16$ and $p = 9$, and the requested difference is $16 - 9 = 7$.

Note: A theorem in the study of Farey fractions states that if $\frac{a}{p} < \frac{b}{q}$ and $bp - aq = 1$, then the rational number with least denominator between $\frac{a}{p}$ and $\frac{b}{q}$ is $\frac{a+b}{p+q}$.

2003A

18. (B) Note that $n = 100q + r = q + r + 99q$. Hence $q + r$ is divisible by 11 if and only if n is divisible by 11. Since $10,000 \leq n \leq 99,999$, there are

$$\left\lfloor \frac{99999}{11} \right\rfloor - \left\lfloor \frac{9999}{11} \right\rfloor = 9090 - 909 = 8181$$

such numbers.

2018B

19. **Answer (C):** Let d be the next divisor of n after 323. Then $\gcd(d, 323) \neq 1$, because otherwise $n \geq 323d > 323^2 > 100^2 = 10000$, contrary to the fact that n is a 4-digit number. Therefore $d - 323 \geq \gcd(d, 323) > 1$. The prime factorization of 323 is $17 \cdot 19$. Thus the next divisor of n is at least $323 + 17 = 340 = 17 \cdot 20$. Indeed, 340 will be the next number in Mary's list when $n = 17 \cdot 19 \cdot 20 = 6460$.

2017A

20. **Answer (E):** Let $u = \log_b a$. Because $u^{2017} = 2017u$, either $u = 0$ or $u = \pm \sqrt[2016]{2017}$. If $u = 0$, then $a = 1$ and b can be any integer from 2 to 200. If $u = \pm \sqrt[2016]{2017}$, then $a = b^{\pm \sqrt[2016]{2017}}$, where again b can be any integer from 2 to 200. Therefore there are $3 \cdot 199 = 597$ such ordered pairs.