UNIT 12 QUESTIONS 16-20

ALGEBRA

2004A

16. **(B)** The given expression is defined if and only if

$$\log_{2003}(\log_{2002}(\log_{2001} x)) > 0,$$

that is, if and only if

$$\log_{2002}(\log_{2001} x) > 2003^0 = 1.$$

This inequality in turn is satisfied if and only if

$$\log_{2001} x > 2002,$$

that is, if and only if $x > 2001^{2002}$.

2003B 17. (D) We have

$$1 = \log(xy^3) = \log x + 3\log y \quad \text{and} \quad 1 = \log(x^2y) = 2\log x + \log y.$$

Solving yields $\log x = \frac{2}{5}$ and $\log y = \frac{1}{5}$. Thus

$$\log(xy) = \log x + \log y = \frac{3}{5}.$$

OR

The given equations imply that $xy^3 = 10 = x^2y$. Thus

$$y = \frac{10}{x^2} \quad \text{and} \quad x\left(\frac{10}{x^2}\right)^3 = 10.$$

It follows that $x = 10^{2/5}$ and $y = 10^{1/5}$, so $\log(xy) = \log(10^{3/5}) = 3/5$.

OR

Since $\log(xy^3) = \log(x^2y)$, we have $xy^3 = x^2y$, so $x = y^2$. Hence

$$1 = \log(xy^3) = \log(y^5) = 5\log y, \quad \text{and} \quad \log y = \frac{1}{5}.$$

So $\log(xy) = \log(y^3) = 3/5$.

2003B 18. (B) We have

$$11y^{13} = 7x^5 = 7(a^cb^d)^5 = 7a^{5c}b^{5d}.$$

The minimum value of x is obtained when neither x nor y contains prime factors other than 7 and 11. Therefore we may assume that $a = 7$ and $b = 11$, so $x = 7^c11^d$ and $7x^5 = 7^{5c+1}11^{5d}$. Letting $y = 7^m11^n$ we obtain $11y^{13} = 7^{13m}11^{13n+1}$. Hence $7^{5c+1}11^{5d} = 7^{13m}11^{13n+1}$. The smallest positive integer solutions are $c = 5$, $d = 8$, $m = 2$, and $n = 3$. Thus $a + b + c + d = 7 + 11 + 5 + 8 = 31$.

2002B

19. **(D)** Adding the given equations gives $2(ab+bc+ca) = 484$, so $ab+bc+ca = 242$. Subtracting from this each of the given equations yields $bc = 90$, $ca = 80$, and $ab = 72$. It follows that $a^2b^2c^2 = 90 \cdot 80 \cdot 72 = 720^2$. Since $abc > 0$, we have $abc = 720$.

2008A

19. **Answer (C):** Each term in the expansion has the form x^{a+b+c}, where $0 \le a \le 27$, $0 \le b \le 14$, and $0 \le c \le 14$. There are $(14+1)^2 = 225$ possible combinations of values for b and c, and for every combination except $(b,c) = (0,0)$, there is a unique a with $a+b+c = 28$. Thus the coefficient of x^{28} is 224.

OR

Let $P(x) = (1 + x + x^2 + \cdots + x^{14})^2 = 1 + r_1x + r_2x^2 + \cdots + r_{28}x^{28}$ and $Q(x) = 1 + x + x^2 + \cdots + x^{27}$. The coefficient of x^{28} in the product $P(x)Q(x)$ is $r_1 + r_2 + \cdots + r_{28} = P(1) - 1 = 15^2 - 1 = 224$.

2011A

19. **Answer (B):** For $0 < x \le 100$, the nearest lattice point directly above the line $y = \frac{1}{2}x + 2$ is $\left(x, \frac{1}{2}x + 3\right)$ if x is even and $\left(x, \frac{1}{2}x + \frac{5}{2}\right)$ if x is odd. The slope of the line that contains this point and $(0,2)$ is $\frac{1}{2} + \frac{1}{x}$ if x is even and $\frac{1}{2} + \frac{1}{2x}$ if x is odd. The minimum value of the slope is $\frac{51}{100}$ if x is even and $\frac{50}{99}$ if x is odd. Therefore the line $y = mx + 2$ contains no lattice point with $0 < x \le 100$ for $\frac{1}{2} < m < \frac{50}{99}$.

1999

20. **(E)** For $n \geq 3$,

$$a_n = \frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}.$$

Thus $(n-1)a_n = a_1 + a_2 + \cdots + a_{n-1}$. It follows that

$$a_{n+1} = \frac{a_1 + a_2 + \cdots + a_{n-1} + a_n}{n} = \frac{(n-1) \cdot a_n + a_n}{n} = a_n,$$

for $n \geq 3$. Since $a_9 = 99$ and $a_1 = 19$, it follows that

$$99 = a_3 = \frac{19 + a_2}{2},$$

and hence that $a_2 = 179$. (The sequ is $19, 179, 99, 99, \ldots$.)

2002A

20. **(C)** Since $0.\overline{ab} = \frac{ab}{99}$, the denominator must be a factor of $99 = 3^2 \cdot 11$. The factors of 99 are 1, 3, 9, 11, 33, and 99. Since a and b are not both nine, the denominator cannot be 1. By choosing a and b appropriately, we can make fractions with each of the other denominators.

2005B 20. **(C)** Note that the sum of the elements in the set is 8. Let $x = a+b+c+d$, so $e+f+g+h = 8-x$. Then

$$(a+b+c+d)^2 + (e+f+g+h)^2 = x^2 + (8-x)^2$$
$$= 2x^2 - 16x + 64 = 2(x-4)^2 + 32 \geq 32.$$

The value of 32 can be attained if and only if $x = 4$. However, it may be assumed without loss of generality that $a = 13$, and no choice of b, c, and d gives a total of 4 for x. Thus $(x-4)^2 \geq 1$, and

$$(a+b+c+d)^2 + (e+f+g+h)^2 = 2(x-4)^2 + 32 \geq 34.$$

A total of 34 can be attained by letting a, b, c, and d be distinct elements in the set $\{-7, -5, 2, 13\}$.

2014B 20. **Answer (B):** The domain of $\log_{10}(x-40) + \log_{10}(60-x)$ is $40 < x < 60$. Within this domain, the inequality $\log_{10}(x-40)+\log_{10}(60-x) < 2$ is equivalent to each of the following: $\log_{10}((x-40)(60-x)) < 2$, $(x-40)(60-x) < 10^2 = 100$, $x^2 - 100x + 2500 > 0$, and $(x-50)^2 > 0$. The last inequality is true for all $x \neq 50$. Thus the integer solutions to the original inequality are $41, 42, \ldots, 49, 51, 52, \ldots, 59$, and their number is 18.