UNIT 10 QUESTIONS 16-20

PROBABILITY

2002A

16. **(A)** There are ten ways for Tina to select a pair of numbers. The sums 9, 8, 4, and 3 can be obtained in just one way, and the sums 7, 6, and 5 can each be obtained in two ways. The probability for each of Sergio's choices is $1/10$. Considering his selections in decreasing order, the total probability of Sergio's choice being greater is

$$\left(\frac{1}{10}\right)\left(1+\frac{9}{10}+\frac{8}{10}+\frac{6}{10}+\frac{4}{10}+\frac{2}{10}+\frac{1}{10}+0+0+0\right)=\frac{2}{5}.$$

2002B 16. **(C)** The product will be a multiple of 3 if and only if at least one of the two rolls is a 3 or a 6. The probability that Juan rolls 3 or 6 is $2/8 = 1/4$. The probability that Juan does not roll 3 or 6, but Amal does is $(3/4)(1/3) = 1/4$. Thus, the probability that the product of the rolls is a multiple of 3 is

$$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

2010A 16. **Answer (B):** The probability that Bernardo picks a 9 is $\frac{3}{9} = \frac{1}{3}$. In this case, his three-digit number will begin with a 9 and will be larger than Silvia's three-digit number.

If Bernardo does not pick a 9, then Bernardo and Silvia will form the same number with probability

$$\frac{1}{\binom{8}{3}} = \frac{1}{56}.$$

If they do not form the same number then Bernardo's number will be larger $\frac{1}{2}$ of the time.

Hence the probability is

$$\frac{1}{3} + \frac{2}{3} \cdot \frac{1}{2}(1 - \frac{1}{56}) = \frac{111}{168} = \frac{37}{56}.$$

2010B

16. **Answer (E):** Let $N = abc + ab + a = a(bc + b + 1)$. If a is divisible by 3, then N is divisible by 3. Note that 2010 is divisible by 3, so the probability that a is divisible by 3 is $\frac{1}{3}$.

If a is not divisible by 3 then N is divisible by 3 if $bc + b + 1$ is divisible by 3. Define b_0 and b_1 so that $b = 3b_0 + b_1$ is an integer and b_1 is equal to 0, 1, or 2. Note that each possible value of b_1 is equally likely. Similarly define c_0 and c_1. Then

$$\begin{aligned} bc + b + 1 &= (3b_0 + b_1)(3c_0 + c_1) + 3b_0 + b_1 + 1 \\ &= 3(3b_0c_0 + c_0b_1 + c_1b_0 + b_0) + b_1c_1 + b_1 + 1. \end{aligned}$$

Hence $bc + b + 1$ is divisible by 3 if and only if $b_1 = 1$ and $c_1 = 1$, or $b_1 = 2$ and $c_1 = 0$. The probability of this occurrence is $\frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} = \frac{2}{9}$.

Therefore the requested probability is $\frac{1}{3} + \frac{2}{3} \cdot \frac{2}{9} = \frac{13}{27}$.

2017B

16. **Answer (B):** There are $\lfloor \frac{21}{2} \rfloor + \lfloor \frac{21}{4} \rfloor + \lfloor \frac{21}{8} \rfloor + \lfloor \frac{21}{16} \rfloor = 10+5+2+1 = 18$ powers of 2 in the prime factorization of 21!. Thus $21! = 2^{18}k$, where k is odd. A divisor of 21! must be of the form 2^ib where $0 \le i \le 18$ and b is a divisor of k. For each choice of b, there is one odd divisor of 21! and 18 even divisors. Therefore the probability that a randomly chosen divisor is odd is $\frac{1}{19}$. In fact, $21! = 2^{18} \cdot 3^9 \cdot 5^4 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19$, so it has $19 \cdot 10 \cdot 5 \cdot 4 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 60{,}800$ positive integer divisors, of which $10 \cdot 5 \cdot 4 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 3{,}200$ are odd.

2006B

17. **(C)** On each die the probability of rolling k, for $1 \le k \le 6$, is

$$\frac{k}{1+2+3+4+5+6} = \frac{k}{21}.$$

There are six ways of rolling a total of 7 on the two dice, represented by the ordered pairs $(1,6)$, $(2,5)$, $(3,4)$, $(4,3)$, $(5,2)$, and $(6,1)$. Thus the probability of rolling a total of 7 is

$$\frac{1\cdot 6 + 2\cdot 5 + 3\cdot 4 + 4\cdot 3 + 5\cdot 2 + 6\cdot 1}{21^2} = \frac{56}{21^2} = \frac{8}{63}.$$

2015A

17. **Answer (A):** There are $2^8 = 256$ equally likely outcomes of the coin tosses. Classify the possible arrangements around the table according to the number of heads flipped. There is 1 possibility with no heads, and there are 8 possibilities with exactly one head. There are $\binom{8}{2} = 28$ possibilities with exactly two heads, 8 of which have two adjacent heads. There are $\binom{8}{3} = 56$ possibilities with exactly three heads, of which 8 have three adjacent heads and $8 \cdot 4$ have exactly two adjacent heads (8 possibilities to place the two adjacent heads and 4 possibilities to place the third head). Finally, there are 2 possibilities using exactly four heads where no two of them are adjacent (heads and tails must alternate). There cannot be more than four heads without two of them being adjacent. Therefore there are $1 + 8 + (28 - 8) + (56 - 8 - 32) + 2 = 47$ possibilities with no adjacent heads, and the probability is $\frac{47}{256}$.

2015B

17. **Answer (D):** The probability of exactly two heads is $\binom{n}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^{n-2}$, and this must equal the probability of three heads, $\binom{n}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^{n-3}$. This results in the equation

$$\frac{n(n-1)}{2} \cdot \frac{3}{4} = \frac{n(n-1)(n-2)}{6} \cdot \frac{1}{4} \qquad \text{or} \qquad \frac{3}{8} = \frac{n-2}{24}.$$

Therefore $n = 11$.

2017B

17. **Answer (D):** Let p be the probability of heads. To win Game A requires that all three tosses be heads, which occurs with probability p^3, or all three tosses be tails, which occurs with probability $(1-p)^3$. To win Game B requires that the first two tosses be the same, the probability of which is $p^2+(1-p)^2$, and that the last two tosses be the same, which occurs with the same probability. Therefore the probability of winning Game A minus the probability of winning Game B is

$$\left(p^3+(1-p)^3\right)-\left(p^2+(1-p)^2\right)^2.$$

As $p=\frac{2}{3}$, this gives

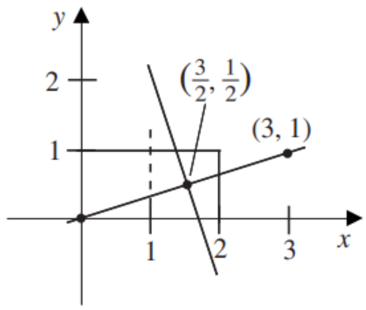
$$\left(\left(\frac{2}{3}\right)^3+\left(\frac{1}{3}\right)^3\right)-\left(\left(\frac{2}{3}\right)^2+\left(\frac{1}{3}\right)^2\right)^2=\frac{1}{3}-\frac{25}{81}=\frac{2}{81}.$$

Thus the probability of winning Game A is $\frac{2}{81}$ greater than the probability of winning Game B.

Note: Expanding and then factoring the general expression above for the probability of winning Game A minus the probability of winning Game B yields $p(1-p)(2p-1)^2$. This value is always nonnegative, so the player should never choose Game B. It equals 0 if and only if $p=0$, $\frac{1}{2}$, or 1. It is maximized when $p=\frac{2\pm\sqrt{2}}{4}$, which is about 85% or 15%, and in this case winning Game A is 6.25 percentage points more likely than winning Game B.

2002B

18. **(C)** The area of the rectangular region is 2. Hence the probability that P is closer to $(0,0)$ than it is to $(3,1)$ is half the area of the trapezoid bounded by the lines $y=1$, the x- and y- axes, and the perpendicular bisector of the segment joining $(0,0)$ and $(3,1)$. The perpendicular bisector goes through the point $(3/2, 1/2)$, which is the center of the square whose vertices are $(1,0), (2,0), (2,1)$, and Hence, the line cuts the square into two quadrilaterals of equal area $1/2$. Thus the area of the trapezoid is $3/2$ and the probability is $3/4$.



2010B

18. **Answer (C):** Let A denote the frog's starting point, and let P, Q, and B denote its positions after the first, second, and third jumps, respectively. Introduce a coordinate system with P at $(0,0)$, Q at $(1,0)$, A at $(\cos\alpha, \sin\alpha)$, and B at $(1+\cos\beta, \sin\beta)$. It may be assumed that $0 \le \alpha \le \pi$ and $0 \le \beta \le 2\pi$. For $\alpha = 0$, the required condition is met for all values of β. For $\alpha = \pi$, the required condition is met only if $\beta = \pi$. For $0 < \alpha < \pi$, $AB = 1$ if and only if $\beta = \alpha$ or $\beta = \pi$, and the required condition is met if and only if $\alpha \le \beta \le \pi$. In the $\alpha\beta$-plane, the rectangle $0 \le \alpha \le \pi$, $0 \le \beta \le 2\pi$ has area $2\pi^2$. The triangle $0 \le \alpha \le \pi$, $\alpha \le \beta \le \pi$ has area $\frac{\pi^2}{2}$, so the requested probability is $\frac{1}{4}$.

2010A

19. **Answer (A):** If Isabella reaches the k^{th} box, she will draw a white marble from it with probability $\frac{k}{k+1}$. For $n \ge 2$, the probability that she will draw white marbles from each of the first $n-1$ boxes is

$$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{n-1}{n} = \frac{1}{n},$$

so the probability that she will draw her first red marble from the n^{th} box is $P(n) = \frac{1}{n(n+1)}$. The condition $P(n) < 1/2010$ is equivalent to $n^2+n-2010 > 0$, from which $n > \frac{1}{2}(-1+\sqrt{8041})$ and $(2n+1)^2 > 8041$. The smallest positive odd integer whose square exceeds 8041 is 91, and the corresponding value of n is 45.

2016A 19. **Answer (B):** Jerry arrives at 4 for the first time after an even number of tosses. Because Jerry tosses 8 coins, he arrives at 4 for the first time after either

4, 6, or 8 tosses. If Jerry arrives at 4 for the first time after 4 tosses, then he must have tossed HHHH. The probability of this occurring is $\frac{1}{16}$. If Jerry arrives at 4 for the first time after 6 tosses, he must have tossed 5 heads and 1 tail among the 6 tosses, and the 1 tail must have come among the first 4 tosses. Thus, there are 4 possible sequences of valid tosses, each with probability $\frac{1}{64}$, for a total of $\frac{1}{16}$. If Jerry arrives at 4 for the first time after 8 tosses, then he must have tossed 6 heads and 2 tails among the 8 tosses. Both tails must occur among the first 6 tosses; otherwise Jerry would have already reached 4 before the 8^{th} toss. Further, at least 1 tail must occur in the first 4 tosses; otherwise Jerry would have already reached 4 after the 4^{th} toss. Therefore there are $\binom{6}{2} - 1 = 14$ sequences for which Jerry first arrives at 4 after 8 tosses, each with probability $\frac{1}{256}$, for a total of $\frac{14}{256} = \frac{7}{128}$. Thus the probability that Jerry reaches 4 at some time during the process is $\frac{1}{16} + \frac{1}{16} + \frac{7}{128} = \frac{23}{128}$. The requested sum is $23 + 128 = 151$.

OR

Count the sequences of 8 heads or tails that result in Jerry arriving at 4. Any sequence with T appearing fewer than 3 times results in Jerry reaching 4. There are $\binom{8}{0} + \binom{8}{1} + \binom{8}{2} = 1 + 8 + 28 = 37$ such sequences. If Jerry's sequence contains exactly 3 Ts, then he reaches 4 only if he does so before getting his second T. As a result, Jerry can get at most one T in his first 5 tosses. This happens if the first 4 tosses are H and there is exactly one H in the last 4 tosses, or there is one T within the first 4 tosses followed by the remaining 5 Hs, accounting for $4 + 4 = 8$ ways for Jerry to get to 4 with exactly 3 Ts. Finally, the only way for Jerry to get to 4 by tossing exactly 4 Ts is HHHHTTTT. Jerry cannot get to 4 by tossing fewer than 4 Hs. Thus there are $37 + 8 + 1 = 46$ sequences where he reaches 4, out of $2^8 = 256$ equally likely ways to toss the coin 8 times. The required probability is $\frac{46}{256} = \frac{23}{128}$, and the requested sum is $23 + 128 = 151$.

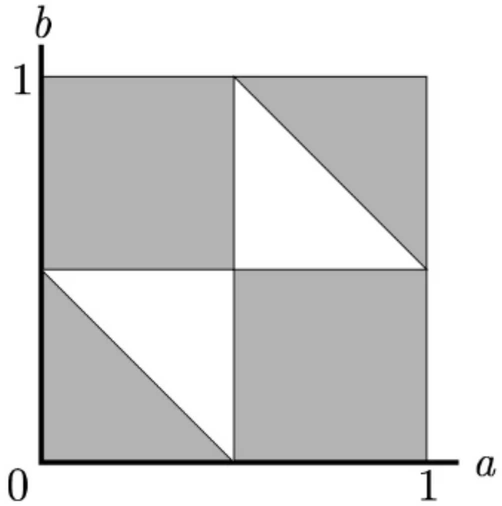
2016B 19. **Answer (B):** The probability that a flipper obtains his first head on the n^{th} flip is $(\frac{1}{2})^n$, because the sequence of outcomes must be exactly TT ... TH, with $n - 1$ Ts. Therefore the probability that all of them obtain their first heads on the n^{th} flip is $\left((\frac{1}{2})^n\right)^3 = (\frac{1}{8})^n$. The probability that all three flip their coins the same number of times is computed by summing an infinite geometric series:

$$\left(\frac{1}{8}\right)^1 + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^3 + \cdots = \frac{\frac{1}{8}}{1 - \frac{1}{8}} = \frac{1}{7}.$$

2004A

20. **(E)** The conditions under which $A + B = C$ are as follows.
(i) If $a + b < 1/2$, then $A = B = C = 0$.
(ii) If $a \geq 1/2$ and $b < 1/2$, then $B = 0$ and $A = C = 1$.
(iii) If $a < 1/2$ and $b \geq 1/2$, then $A = 0$ and $B = C = 1$.
(iv) If $a + b \geq 3/2$, then $A = B = 1$ and $C = 2$.

These conditions correspond to the shaded regions of the graph shown. The combined area of those regions is $3/4$, and the area of the entire square is 1, so the requested probability is $3/4$.



2004B

20. **(B)** If the orientation of the cube is fixed, there are $2^6 = 64$ possible arrangements of colors on the faces. There are

$$2\binom{6}{6} = 2$$

arrangements in which all six faces are the same color and

$$2\binom{6}{5} = 12$$

arrangements in which exactly five faces have the same color. In each of these cases the cube can be placed so that the four vertical faces have the same color. The only other suitable arrangements have four faces of one color, with the other color on a pair of opposing faces. Since there are three pairs of opposing faces, there are $2(3) = 6$ such arrangements. The total number of suitable arrangements is therefore $2 + 12 + 6 = 20$, and the probability is $20/64 = 5/16$.

2006B

20. **(C)** The given condition is equivalent to $\lfloor \log_{10} x \rfloor = \lfloor \log_{10} 4x \rfloor$. Thus the condition holds if and only if

$$n \le \log_{10} x < \log_{10} 4x < n + 1$$

for some negative integer n. Equivalently,

$$10^n \le x < 4x < 10^{n+1}.$$

This inequality is true if and only if

$$10^n \le x < \frac{10^{n+1}}{4}.$$

Hence in each interval $\left[10^n, 10^{n+1}\right)$, the given condition holds with probability

$$\frac{(10^{n+1}/4) - 10^n}{10^{n+1} - 10^n} = \frac{10^n((10/4) - 1)}{10^n(10 - 1)} = \frac{1}{6}.$$

Because each number in $(0, 1)$ belongs to a unique interval $\left[10^n, 10^{n+1}\right)$ and the probability is the same on each interval, the required probability is also $1/6$.

2017B

20. **Answer (D):** The set of all possible ordered pairs (x, y) is bounded by the unit square in the coordinate plane with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$. For each positive integer n, $\lfloor \log_2 x \rfloor = \lfloor \log_2 y \rfloor = -n$ if and only if $\frac{1}{2^n} \le x < \frac{1}{2^{n-1}}$ and $\frac{1}{2^n} \le y < \frac{1}{2^{n-1}}$. Thus the set of ordered pairs (x, y) such that $\lfloor \log_2 x \rfloor = \lfloor \log_2 y \rfloor = -n$ is bounded by a square with side length $\frac{1}{2^n}$ and therefore area $\frac{1}{4^n}$. The union of these squares over all positive integers n has area

$$\sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3},$$

and therefore the requested probability is $\frac{1}{3}$. (It is also clear from the diagram that one third of the square is shaded.)

$$\sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3},$$

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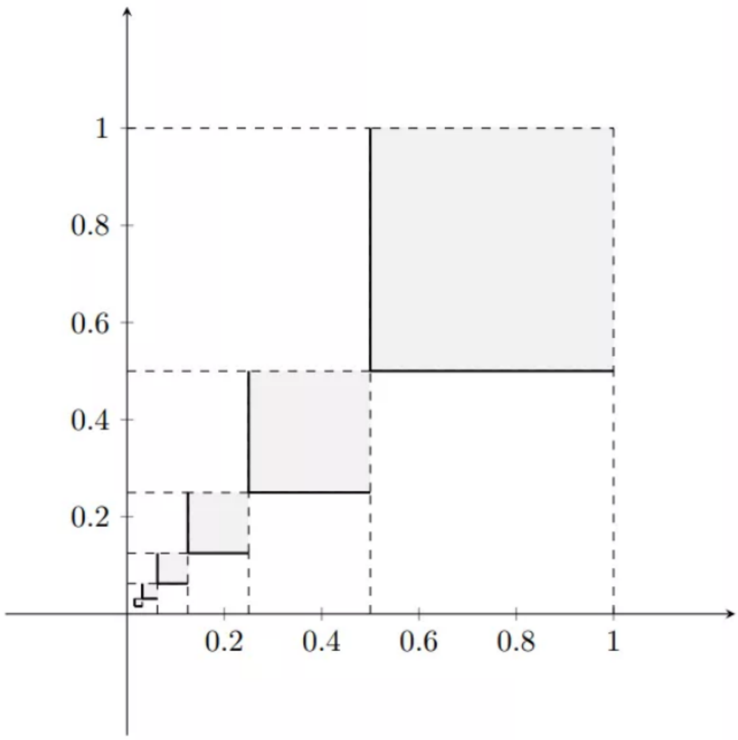

OR

The problem can be modeled with Xerxes and Yolanda each repeatedly flipping a fair coin to determine the binary (base two "decimal") expansions of x and y, respectively. If Xerxes flips a head, he writes down a 0 as the next binary digit; if he flips a tail, he writes down a 1. Yolanda does the same. Then $\lfloor \log_2 x \rfloor = \lfloor \log_2 y \rfloor$ if and only if the first time that either of them flips a tail, so does the other. There are three equally likely outcomes: tail-tail, tail-head, and head-tail. Therefore the requested probability is $\frac{1}{3}$.