UNIT 1 EXERCISES 16-20

2D GEOMETRY

2003B

2005A

2006A

2017A

2001

2003A

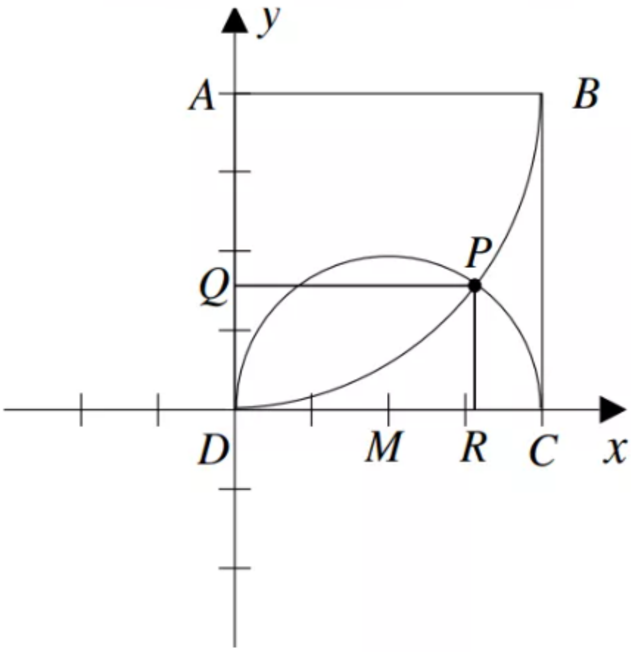
17. (B) Place an xy-coordinate system with origin at D and points C and A on the positive x- and y-axes, respectively. Then the circle centered at M has equation

$$(x-2)^2+y^2=4,$$

and the circle centered at A has equation

$$x^2+(y-4)^2=16.$$

Solving these equations for the coordinates of P gives $x=16/5$ and $y=8/5$, so the answer is $16/5$.



OR

We have $AP=AD=4$ and $PM=MD=2$, so $\triangle ADM$ is congruent to $\triangle APM$, and $\angle APM$ is a right angle. Draw $\overline{PQ}$ and $\overline{PR}$ perpendicular to $\overline{AD}$ and $\overline{CD}$, respectively. Note that $\angle APQ$ and $\angle MPR$ are both complements of $\angle QPM$. Thus $\triangle APQ$ is similar to $\triangle MPR$, and

$$\frac{AQ}{MR}=\frac{AP}{MP}=\frac{4}{2}=2.$$

Let $MR=x$. Then $AQ=2x$, $PR=QD=4-2x$, and $PQ=RD=x+2$. Therefore

$$2=\frac{AQ}{MR}=\frac{PQ}{PR}=\frac{x+2}{4-2x},$$

so $x=6/5$ and $PQ=6/5+2=16/5$.

OR

Let $\angle MAD=\alpha$. Then

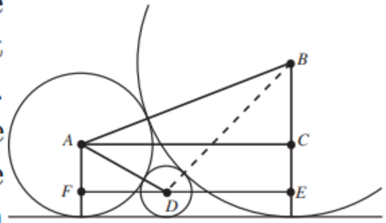
$$PQ=(PA)\sin(\angle PAQ)=4\sin(2\alpha)=8\sin\alpha\cos\alpha=8\left(\frac{2}{\sqrt{20}}\right)\left(\frac{4}{\sqrt{20}}\right)=\frac{16}{5}.$$

2006A

2016B 17. **Answer (D):** Let $x = BH$. Then $CH = 8 - x$ and $AH^2 = 7^2 - x^2 = 9^2 - (8-x)^2$, so $x = 2$ and $AH = \sqrt{45}$. By the Angle Bisector Theorem in $\triangle ACH$, $\frac{AP}{PH} = \frac{CA}{CH} = \frac{9}{6}$, so $AP = \frac{3}{5}AH$. Similarly, by the Angle Bisector Theorem in $\triangle ABH$, $\frac{AQ}{QH} = \frac{BA}{BH} = \frac{7}{2}$, so $AQ = \frac{7}{9}AH$. Then $PQ = AQ - AP = (\frac{7}{9} - \frac{3}{5})AH = \frac{8}{45}\sqrt{45} = \frac{8}{15}\sqrt{5}$.

2001

18. **(D)** Let C be the intersection of the horizontal line through A and the vertical line through B. In right triangle ABC, we have $BC = 3$ and $AB = 5$, so $AC = 4$. Let x be the radius of the third circle, and D be the center. Let E and F be the points of intersection of the horizontal line through D with the vertical lines through B and A, respectively, as shown.



In $\triangle BED$ we have $BD = 4 + x$ and $BE = 4 - x$, so

$$DE^2 = (4 + x)^2 - (4 - x)^2 = 16x,$$

and $DE = 4\sqrt{x}$. In $\triangle ADF$ we have $AD = 1 + x$ and $AF = 1 - x$, so

$$FD^2 = (1 + x)^2 - (1 - x)^2 = 4x,$$

and $FD = 2\sqrt{x}$. Hence,

$$4 = AC = FD + DE = 2\sqrt{x} + 4\sqrt{x} = 6\sqrt{x}$$

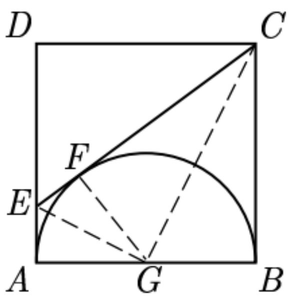
and $\sqrt{x} = \frac{2}{3}$, which implies $x = \frac{4}{9}$.

2004A

18. **(D)** Let F be the point at which $\overline{CE}$ is tangent to the semicircle, and let G be the midpoint of $\overline{AB}$. Because $\overline{CF}$ and $\overline{CB}$ are both tangents to the semicircle, $CF = CB = 2$. Similarly, $EA = EF$. Let $x = AE$. The Pythagorean Theorem applied to $\triangle CDE$ gives

$$(2 - x)^2 + 2^2 = (2 + x)^2.$$

It follows that $x = 1/2$ and $CE = 2 + x = 5/2$.



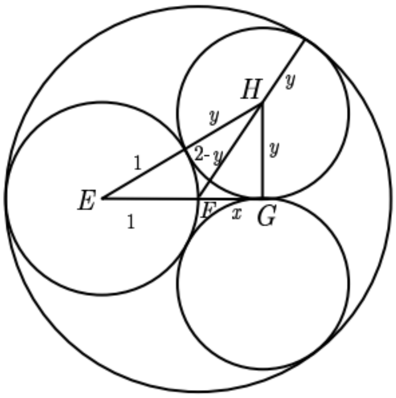
2004A 19. **(D)** Let E, H, and F be the centers of circles A, B, and D, respectively, and let G be the point of tangency of circles B and C. Let $x = FG$ and $y = GH$. Since the center of circle D lies on circle A and the circles have a common point of tangency, the radius of circle D is 2, which is the diameter of circle A. Applying the Pythagorean Theorem to right triangles EGH and FGH gives

$$(1+y)^2 = (1+x)^2 + y^2 \quad \text{and} \quad (2-y)^2 = x^2 + y^2,$$

from which it follows that

$$y = x + \frac{x^2}{2} \quad \text{and} \quad y = 1 - \frac{x^2}{4}.$$

The solutions of this system are $(x, y) = (2/3, 8/9)$ and $(x, y) = (-2, 0)$. The radius of circle B is the positive solution for y, which is $8/9$.



2006A 19. **(E)** The slope of the line l containing the centers of the circles is $5/12 = \tan\theta$, where θ is the acute angle between the x-axis and line l. The equation of line l is $y - 4 = (5/12)(x - 2)$. This line and the two common external tangents are concurrent. Because one of these tangents is the x-axis, the point of intersection is the x-intercept of line l, which is $(-38/5, 0)$. The acute angle between the x-axis and the other tangent is 2θ, so the slope of that tangent is

$$\tan 2\theta = 2 \cdot \frac{5/12}{1-(5/12)^2} = \frac{120}{119}.$$

Thus the equation of that tangent is $y = (120/119)\,(x + (38/5))$, and

$$b = \frac{120}{119} \cdot \frac{38}{5} = \frac{912}{119}.$$