UNIT 5 EXERCISES 11-15

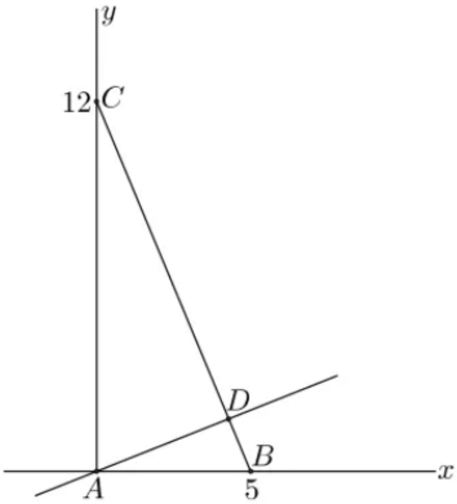
CO-ORD

2011B

11. **Answer (B):** Because $AB = 1$, the smallest number of jumps is at least 2. The perpendicular bisector of $\overline{AB}$ is the line with equation $x = \frac{1}{2}$, which has no points with integer coordinates, so 2 jumps are not possible. A sequence of 3 jumps is possible; one such sequence is $(0, 0)$ to $(3, 4)$ to $(6, 0)$ to $(1, 0)$.

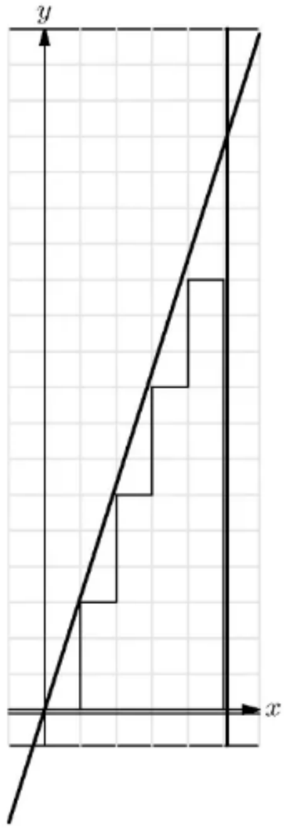
2015B

11. **Answer (E):** Label the vertices of the triangle $A = (0, 0)$, $B = (5, 0)$, and $C = (0, 12)$. By the Pythagorean Theorem $BC = 13$. Two altitudes are 5 and 12. Let $\overline{AD}$ be the third altitude. The area of this triangle is 30, so $\frac{1}{2} \cdot AD \cdot BC = 30$. Therefore $AD = \frac{2 \cdot 30}{BC} = \frac{60}{13}$. The sum of the lengths of the altitudes is $5 + 12 + \frac{60}{13} = \frac{281}{13}$.



2016B

11. **Answer (D):** Note that $3 < \pi < 4$, $6 < 2\pi < 7$, $9 < 3\pi < 10$, and $12 < 4\pi < 13$. Therefore there are 3 1-by-1 squares of the desired type in the strip $1 \le x \le 2$, 6 1-by-1 squares in the strip $2 \le x \le 3$, 9 1-by-1 squares in the strip $3 \le x \le 4$, and 12 1-by-1 squares in the strip $4 \le x \le 5$. Furthermore there are 2 2-by-2 squares in the strip $1 \le x \le 3$, 5 2-by-2 squares in the strip $2 \le x \le 4$, and 8 2-by-2 squares in the strip $3 \le x \le 5$. There is 1 3-by-3 square in the strip $1 \le x \le 4$, and there are 4 3-by-3 squares in the strip $2 \le x \le 5$. There are no 4-by-4 or larger squares. Thus in all there are $3+6+9+12+2+5+8+1+4 = 50$ squares of the desired type within the given region.



1999

12. **(C)** The x-coordinates of the intersection points are precisely the zeros of the polynomial $p(x) - q(x)$. This polynomial has degree at most three, so it has at most three zeros. Hence, the graphs of the fourth degree polynomial functions intersect at most three times. Finding an example to show that three intersection points can be achieved is left to the reader.

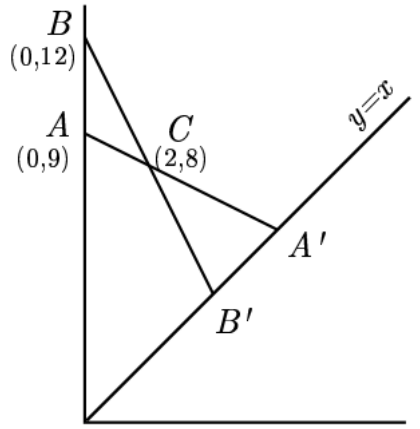
2004A

12. **(B)** Line AC has slope $-\frac{1}{2}$ and y-intercept (0,9), so its equation is

$$y = -\frac{1}{2}x + 9.$$

Since the coordinates of A' satisfy both this equation and $y = x$, it follows that $A' = (6, 6)$. Similarly, line BC has equation $y = -2x + 12$, and $B' = (4, 4)$. Thus

$$A'B' = \sqrt{(6-4)^2 + (6-4)^2} = 2\sqrt{2}.$$



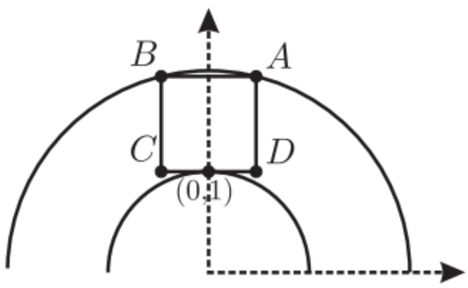
2005A 12. **(D)** The slope of the line is

$$\frac{1000-1}{100-1}=\frac{111}{11},$$

so all points on the line have the form $(1+11t, 1+111t)$. Such a point has integer coordinates if and only if t is an integer, and the point is strictly between A and B if and only if $0 < t < 9$. Thus there are 8 points with the required property.

2012A 12. **Answer (D):**

Suppose by symmetry that $A = (a, b)$ with $a > 0$. Because $ABCD$ is tangent to the circle with equation $x^2 + y^2 = 1$ at $(0, 1)$ and both A and B are on the concentric circle with equation $x^2 + y^2 = 4$, it follows that $B = (-a, b)$. Then the horizontal length of the square is $2a$ and its vertical height is $b-1$. Therefore $2a = b - 1$, or $b = 2a + 1$. Substituting this into the equation $a^2 + b^2 = 4$ leads to the equation $5a^2 + 4a - 3 = 0$. By the quadratic formula, the positive root is $\frac{1}{5}(\sqrt{19} - 2)$, and so the side length $2a$ is $\frac{1}{5}(2\sqrt{19} - 4)$.



2015A 12. **Answer (B):** The y-intercepts of the two parabolas are -2 and 4, respectively, and in order to intersect the x-axis, the first must open upward and the second downward. Because the area of the kite is 12, the x-intercepts of both parabolas must be -2 and 2. Thus $4a-2=0$ so $a=\frac{1}{2}$, and $4-4b=0$ so $b=1$. Therefore $a+b=1.5$.

2001 13. **(E)** The equation of the first parabola can be written in the form

$$y = a(x-h)^2 + k = ax^2 - 2axh + ah^2 + k,$$

and the equation for the second (having the same shape and vertex, but opening in the opposite direction) can be written in the form

$$y = -a(x-h)^2 + k = -ax^2 + 2axh - ah^2 + k.$$

Hence,

$$a+b+c+d+e+f = a+(-2ah)+(ah^2+k)+(-a)+(2ah)+(-ah^2+k) = 2k.$$

OR

The reflection of a point (x, y) about the line $y = k$ is $(x, 2k-y)$. Thus, the equation of the reflected parabola is

$$2k - y = ax^2 + bx + c, \text{ or equivalently, } y = 2k - (ax^2 + bx + c).$$

Hence $a+b+c+d+e+f = 2k$.

2004A 13. **(B)** There are $\binom{9}{2} = 36$ pairs of points in S, and each pair determines a line. However, there are three horizontal, three vertical, and two diagonal lines that pass through three points of S, and these lines are each determined by three different pairs of points in S. Thus the number of distinct lines is $36 - 2 \cdot 8 = 20$.

OR

There are 3 vertical lines, 3 horizontal lines, 3 each with slopes 1 and -1, and 2 each with slopes 2, -2, $1/2$, and $-1/2$, for a total of 20.

2007A 13. **Answer (B):** The point (a, b) is the foot of the perpendicular from $(12, 10)$ to the line $y = -5x + 18$. The perpendicular has slope $\frac{1}{5}$, so its equation is

$$y = 10 + \frac{1}{5}(x - 12) = \frac{1}{5}x + \frac{38}{5}.$$

The x-coordinate at the foot of the perpendicular satisfies the equation

$$\frac{1}{5}x + \frac{38}{5} = -5x + 18,$$

so $x = 2$ and $y = -5 \cdot 2 + 18 = 8$. Thus $(a, b) = (2, 8)$, and $a + b = 10$.

OR

If the mouse is at $(x, y) = (x, 18 - 5x)$, then the square of the distance from the mouse to the cheese is

$$(x - 12)^2 + (8 - 5x)^2 = 26\left(x^2 - 4x + 8\right) = 26\left((x - 2)^2 + 4\right).$$

The value of this expression is smallest when $x = 2$, so the mouse is closest to the cheese at the point $(2, 8)$, and $a + b = 2 + 8 = 10$.

2010A

13. **Answer (C):** When $k = 0$, the graphs of $x^2 + y^2 = 0$ and $xy = 0$ consist of the single point $\{(0,0)\}$ and the union of the two lines $x = 0$ and $y = 0$, respectively; so the two graphs intersect. When $k \neq 0$, the graph of $x^2 + y^2 = k^2$ is a circle of radius k centered at the origin and the graph of $xy = k$ is an equilateral hyperbola centered at the origin. The vertices of the hyperbola, located at $(\pm\sqrt{k}, \pm\sqrt{k})$ if $k > 0$ or at $(\pm\sqrt{-k}, \mp\sqrt{-k})$ if $k < 0$, are the closest points on the graph to the origin. If $|k| \geq 2$, then

$$(\sqrt{|k|})^2 + (\sqrt{|k|})^2 = 2\,|k| \leq k^2,$$

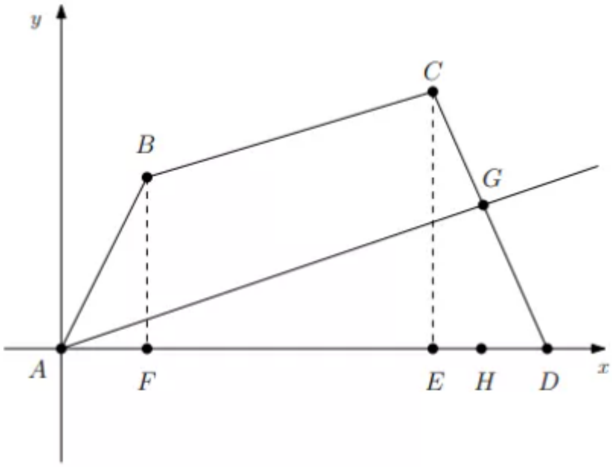
thus the graphs intersect. If $|k| = 1$, then

$$(\sqrt{|k|})^2 + (\sqrt{|k|})^2 = 2 > 1 = k^2,$$

and thus the graphs do not intersect. Thus the graphs do not intersect for $k = 1$ or $k = -1$.

2013A

13. **Answer (B):** Let line AG be the required line, with G on $\overline{CD}$. Divide $ABCD$ into triangle ABF, trapezoid $BCEF$, and triangle CDE, as shown. Their areas are 1, 5, and $\frac{3}{2}$, respectively. Hence the area of $ABCD = \frac{15}{2}$, and the area of triangle $ADG = \frac{15}{4}$. Because $AD = 4$, it follows that $GH = \frac{15}{8} = \frac{r}{s}$. The equation of $\overline{CD}$ is $y = -3(x-4)$, so when $y = \frac{15}{8}$, $x = \frac{p}{q} = \frac{27}{8}$. Therefore $p+q+r+s = 58$.



2008A

14. **Answer (A):** The boundaries of the region are the two pairs of parallel lines

$$(3x-18)+(2y+7)=\pm 3 \quad \text{and} \quad (3x-18)-(2y+7)=\pm 3.$$

These lines intersect at $(6,-2)$, $(6,-5)$, $(5,-\frac{7}{2})$, and $(7,-\frac{7}{2})$. Thus the region is a rhombus whose diagonals have lengths 2 and 3. The area of the rhombus is half the product of the diagonal lengths, which is 3.

2009A 14. **Answer (B):** The line must contain the midpoint of the segment joining $(1,1)$ and $(6m,0)$, which is $\left(\frac{6m+1}{2},\frac{1}{2}\right)$. Thus

$$m=\frac{\frac{1}{2}}{\frac{6m+1}{2}}=\frac{1}{6m+1},$$

from which $0=6m^2+m-1=(3m-1)(2m+1)$. The two possible values of m are $-\frac{1}{2}$ and $\frac{1}{3}$, and their sum is $-\frac{1}{6}$.

If $m=-\frac{1}{2}$ then the triangle with vertices $(0,0)$, $(1,1)$, and $(-3,0)$ is bisected by the line passing through the origin and $(-1,\frac{1}{2})$. Similarly, when $m=\frac{1}{3}$ the triangle with vertices $(0,0)$, $(1,1)$, and $(2,0)$ is bisected by the line passing through the origin and $(\frac{3}{2},\frac{1}{2})$.