UNIT 4 EXERCISES 11-15

TRIANGLES

2013A 12. **Answer (A):** Let the angles of the triangle be $\alpha - \delta$, α, and $\alpha + \delta$. Then $3\alpha = \alpha - \delta + \alpha + \alpha + \delta = 180°$, so $\alpha = 60°$. There are three cases depending on which side is opposite to the $60°$ angle. Suppose that the triangle is ABC with $\angle BAC = 60°$. Let D be the foot of the altitude from C. The triangle CAD is a 30-60-90° triangle, so $AD = \frac{1}{2}AC$ and $CD = \frac{\sqrt{3}}{2}AC$. There are three cases to consider. In each case the Pythagorean Theorem can be used to solve for the unknown side.

If $AB = 5$, $AC = 4$, and $BC = x$, then $AD = 2$, $CD = 2\sqrt{3}$, and $BD = |AB - AD| = 3$. It follows that $x^2 = BC^2 = CD^2 + BD^2 = 21$, so $x = \sqrt{21}$.

If $AB = x$, $AC = 4$, and $BC = 5$, then $AD = 2$, $CD = 2\sqrt{3}$, and $BD = |AB - AD| = |x - 2|$. It follows that $25 = BC^2 = CD^2 + BD^2 = 12 + (x-2)^2$, and the positive solution is $x = 2 + \sqrt{13}$.

If $AB = x$, $AC = 5$, and $BC = 4$, then $AD = \frac{5}{2}$, $CD = \frac{5\sqrt{3}}{2}$, and $BD = |AB - AD| = |x - \frac{5}{2}|$. It follows that $16 = BC^2 = CD^2 + BD^2 = \frac{75}{4} + (x - \frac{5}{2})^2$, which has no solution because $\frac{75}{4} > 16$.

The sum of all possible side lengths is $2 + \sqrt{13} + \sqrt{21}$. The requested sum is $2 + 13 + 21 = 36$.

OR

As in the first solution, there are three cases depending on which side is opposite to the $60°$ angle. In each case, the Law of Cosines can be used to solve for the unknown side. If the unknown side is opposite to the $60°$ angle, then

$$x^2 = 4^2 + 5^2 - 2 \cdot 4 \cdot 5 \cdot \cos(60°) = 21,$$

so $x = \sqrt{21}$.

If the side of length 5 is opposite to the $60°$ angle, then

$$5^2 = x^2 + 4^2 - 2 \cdot 4 \cdot x \cdot \cos(60°) = x^2 - 4x + 16,$$

and the positive solution is $2 + \sqrt{13}$.

If the side of length 4 is opposite to the $60°$ angle, then

$$4^2 = x^2 + 5^2 - 2 \cdot x \cdot 5 \cdot \cos(60°) = x^2 - 5x + 25,$$

which has no real solutions.

The sum of all possible side lengths is $2 + \sqrt{13} + \sqrt{21}$. The requested sum is $2 + 13 + 21 = 36$.

2014B

12. **Answer (B):** Denote a triangle by the string of its side lengths written in nonincreasing order. Then S has at most one equilateral triangle and at most one of the two triangles 442 and 221. The other possible elements of S are 443, 441, 433, 432, 332, 331, and 322. All other strings are excluded by the triangle inequality. Therefore S has at most 9 elements.

2018B

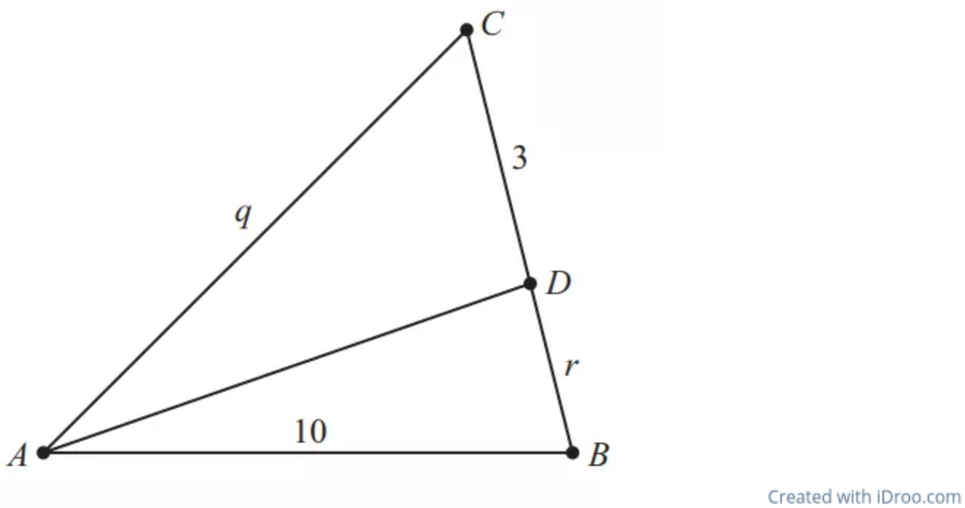
12. **Answer (C):** Let $q = AC$ and $r = BD$. By the Angle Bisector Theorem,

$$\frac{AC}{CD} = \frac{AB}{BD}, \quad \text{which means} \quad \frac{q}{3} = \frac{10}{r}, \quad \text{so} \quad r = \frac{30}{q}.$$

The possible values of AC can be determined by considering the three Triangle Inequalities in $\triangle ABC$.

- $AC + BC > AB$, which means $q + 3 + r > 10$. Substituting for r and simplifying gives $q^2 - 7q + 30 > 0$, which always holds because $q^2 - 7q + 30 = \left(q - \frac{7}{2}\right)^2 + \frac{71}{4}$.
- $BC + AB > AC$, which means $3 + r + 10 > q$. Substituting $r = \frac{30}{q}$, simplifying, and factoring gives $(q - 15)(q + 2) < 0$, which holds if and only if $-2 < q < 15$.
- $AB + AC > BC$, which means $10 + q > 3 + r$. Substituting $r = \frac{30}{q}$, simplifying, and factoring gives $(q + 10)(q - 3) > 0$, which holds if and only if $q > 3$ or $q < -10$.

Combining these inequalities shows that the set of possible values of q is the open interval $(3, 15)$, and the requested sum of the endpoints of the interval is $3 + 15 = 18$.

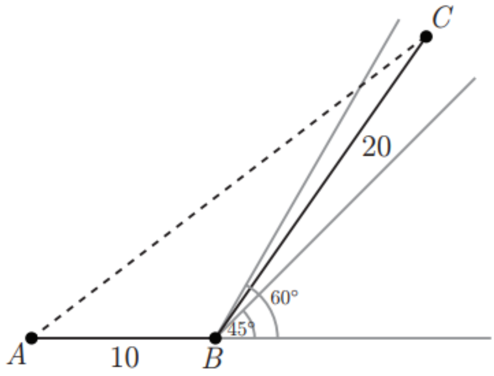


2009A 13. **Answer (D):** By the Law of Cosines,

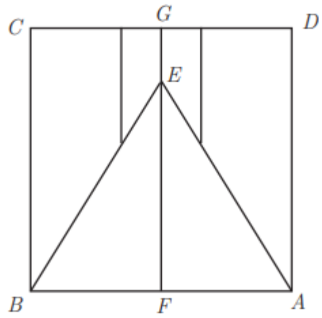
$$AC^2 = AB^2 + BC^2 - 2 \cdot AB \cdot BC \cdot \cos\angle ABC = 500 - 400\cos\angle ABC.$$

Because $\cos\angle ABC$ is between $\cos 120^\circ = -\frac{1}{2}$ and $\cos 135^\circ = -\frac{\sqrt{2}}{2}$, it follows that

$$700 = 500 + 200 \le AC^2 \le 500 + 200\sqrt{2} < 800.$$



2008B



13. **Answer (B):** Draw a line parallel to $\overline{AD}$ through point E, intersecting $\overline{AB}$ at F and intersecting $\overline{CD}$ at G. Triangle AEF is a $30-60-90°$ triangle with hypotenuse $AE = 1$, so $EF = \frac{\sqrt{3}}{2}$. Region R consists of two congruent trapezoids of height $\frac{1}{6}$, shorter base $EG = 1 - \frac{\sqrt{3}}{2}$, and longer base the weighted average

$$\frac{2}{3}EG + \frac{1}{3}AD = \frac{2}{3}\left(1 - \frac{\sqrt{3}}{2}\right) + \frac{1}{3} \cdot 1 = 1 - \frac{\sqrt{3}}{3}.$$

Therefore the area of R is

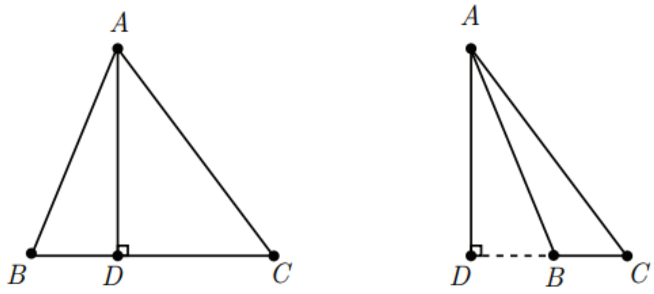
$$2 \cdot \frac{1}{6} \cdot \frac{1}{2}\left(\left(1 - \frac{\sqrt{3}}{2}\right) + \left(1 - \frac{\sqrt{3}}{3}\right)\right) = \frac{1}{6}\left(2 - \frac{5\sqrt{3}}{6}\right) = \frac{12 - 5\sqrt{3}}{36}$$

OR

Place $ABCD$ in a coordinate plane with $B = (0, 0)$, $A = (1, 0)$, and $C = (0, 1)$. Then the equation of the line BE is $y = \sqrt{3}x$, so $E = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, and the point of R closest to B is $(\frac{1}{3}, \frac{\sqrt{3}}{3})$. Thus the region R consists of two congruent trapezoids with height $\frac{1}{6}$ and bases $1 - \frac{\sqrt{3}}{2}$ and $1 - \frac{\sqrt{3}}{3}$. Then proceed as in the first solution.

2009B

13. **Answer (D):** Let D be the foot of the altitude to $\overline{BC}$. Then $BD = \sqrt{13^2 - 12^2} = 5$ and $DC = \sqrt{15^2 - 12^2} = 9$. Thus $BC = BD+DC = 5+9 = 14$ or $BC = DC-BD = 9-5 = 4$. The sum of the two possible values is $14+4 = 18$.



2011A

13. **Answer (B):** The largest pairwise difference is 9, so $w - z = 9$. Let n be either x or y. Because n is between w and z,

$$9 = w - z = (w - n) + (n - z).$$

Therefore the positive differences $w - n$ and $n - z$ must sum to 9. The given pairwise differences that sum to 9 are $3 + 6$ and $4 + 5$. The remaining pairwise difference must be $x - y = 1$.

The second largest pairwise difference is 6, so either $w - y = 6$ or $x - z = 6$. In the first case the set of four numbers may be expressed as $\{w, w-5, w-6, w-9\}$. Hence $4w - 20 = 44$, so $w = 16$. In the second case $w - x = 3$, and the four numbers may be expressed as $\{w, w - 3, w - 4, w - 9\}$. Therefore $4w - 16 = 44$, so $w = 15$. The sum of the possible values for w is $16 + 15 = 31$.

Note: The possible sets of four numbers are $\{16, 11, 10, 7\}$ and $\{15, 12, 11, 6\}$.

2014A 13. **Answer (B):** If each friend rooms alone, then there are $5! = 120$ ways to assign the guests to the rooms. If one pair of friends room together and the others room alone, then there are $\binom{5}{2} = 10$ ways to choose the roommates and then $5 \cdot 4 \cdot 3 \cdot 2 = 120$ ways to assign the rooms to the 4 sets of occupants, for a total of $10 \cdot 120 = 1200$ possible arrangements. The only other possibility is to have two sets of roommates. In this case the roommates can be chosen in $5 \cdot \frac{1}{2}\binom{4}{2} = 15$ ways (choose the solo lodger first), and then there are $5 \cdot 4 \cdot 3 = 60$ ways to assign the rooms, for a total of $15 \cdot 60 = 900$ possibilities. Therefore the answer is $120 + 1200 + 900 = 2220$.

2014B 13. **Answer (C):** There is a triangle with side lengths 1, a, and b if and only if $a > b - 1$. There is a triangle with side lengths $\frac{1}{b}$, $\frac{1}{a}$, and 1 if and only if $\frac{1}{a} > 1 - \frac{1}{b}$, that is, $a < \frac{b}{b-1}$. Therefore there are no such triangles if and only if $b - 1 \geq a \geq \frac{b}{b-1}$. The smallest possible value of b satisfies $b - 1 = \frac{b}{b-1}$, or $b^2 - 3b + 1 = 0$. The solution with $b > 1$ is $\frac{1}{2}(3 + \sqrt{5})$. The corresponding value of a is $\frac{1}{2}(1 + \sqrt{5})$.

2016B 13. **Answer (E):** Let Alice, Bob, and the airplane be located at points A, B, and C, respectively. Let D be the point on the ground directly beneath the airplane, and let h be the airplane's altitude, in miles. Then $\triangle ACD$ and $\triangle BCD$ are $30-60-90^\circ$ right triangles with right angles at D, so $AD = \sqrt{3}h$ and $BD = \frac{h}{\sqrt{3}}$. Then by the Pythagorean Theorem applied to the right triangle on the ground,

$$100 = AB^2 = AD^2 + BD^2 = \left(\sqrt{3}h\right)^2 + \left(\frac{h}{\sqrt{3}}\right)^2 = \frac{10h^2}{3}.$$

Thus $h = \sqrt{30}$, and the closest of the given choices is 5.5.

2004B

14. **(D)** Because $\triangle ABC$, $\triangle NBK$, and $\triangle AMJ$ are similar right triangles whose hypotenuses are in the ratio $13 : 8 : 1$, their areas are in the ratio $169 : 64 : 1$.

The area of $\triangle ABC$ is $\frac{1}{2}(12)(5) = 30$, so the areas of $\triangle NBK$ and$\triangle AMJ$ are $\frac{64}{169}(30)$ and $\frac{1}{169}(30)$, respectively.

Thus the area of pentagon $CMJKN$ is $(1 - \frac{64}{169} - \frac{1}{169})(30) = 240/13$.

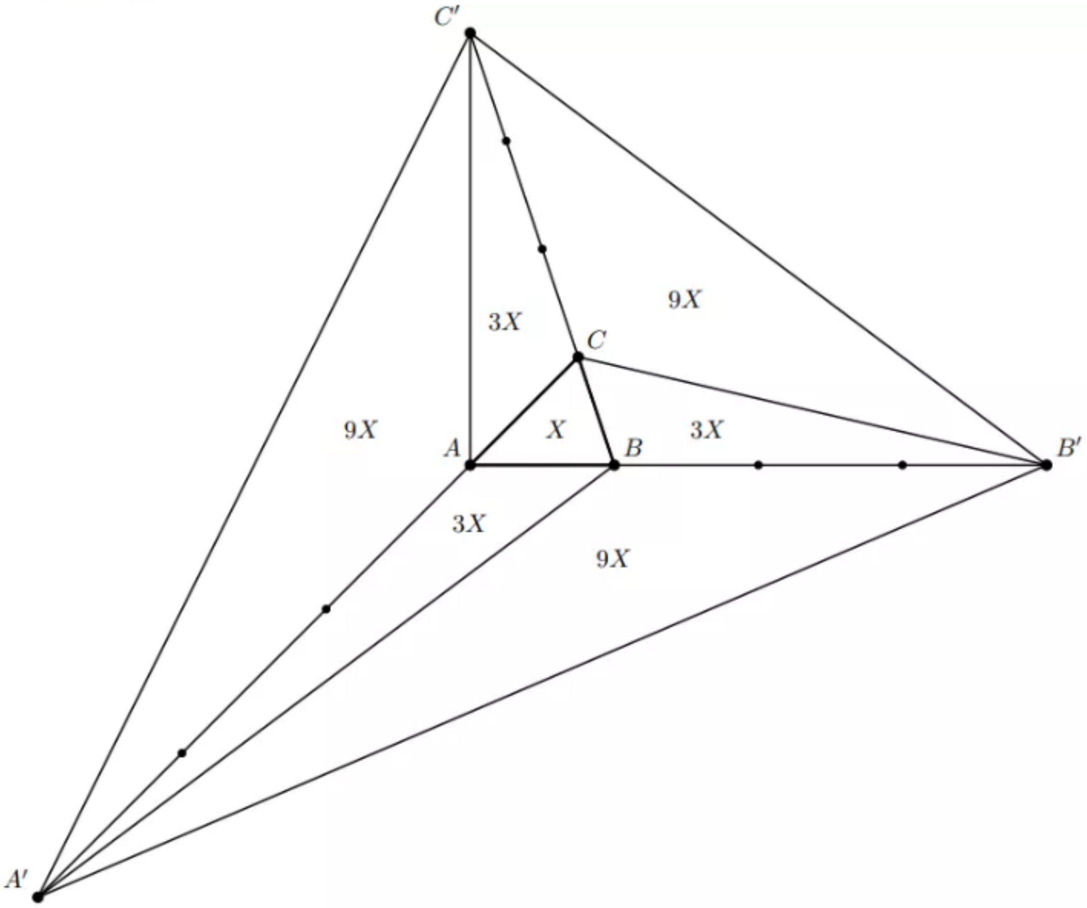
2007B

14. **Answer (D):** Let the side length of $\triangle ABC$ be s. Then the areas of $\triangle APB$, $\triangle BPC$, and $\triangle CPA$ are, respectively, $s/2$, s, and $3s/2$. The area of $\triangle ABC$ is the sum of these, which is $3s$. The area of $\triangle ABC$ may also be expressed as $(\sqrt{3}/4)s^2$, so $3s = (\sqrt{3}/4)s^2$. The unique positive solution for s is $4\sqrt{3}$.

2010A

14. **Answer (B):** By the Angle Bisector Theorem, $8 \cdot BA = 3 \cdot BC$. Thus BA must be a multiple of 3. If $BA = 3$, the triangle is degenerate. If $BA = 6$, then $BC = 16$, and the perimeter is $6 + 16 + 11 = 33$.

2017B 15. **Answer (E):** Draw segments $\overline{CB'}$, $\overline{AC'}$, and $\overline{BA'}$. Let X be the area of $\triangle ABC$. Because $\triangle BB'C$ has a base 3 times as long and the same altitude, its area is $3X$. Similarly, the areas of $\triangle AA'B$ and $\triangle CC'A$ are also $3X$. Furthermore, $\triangle AA'C'$ has 3 times the base and the same height as $\triangle ACC'$, so its area is $9X$. The areas of $\triangle CC'B'$ and $\triangle BB'A'$ are also $9X$ by the same reasoning. Therefore the area of $\triangle A'B'C'$ is $X + 3(3X) + 3(9X) = 37X$, and the requested ratio is $37 : 1$. Note that nothing in this argument requires $\triangle ABC$ to be equilateral.



OR

Let $s = AB$. Applying the Law of Cosines to $\triangle B'BC'$ gives

$$(B'C')^2 = (3s)^2 + (4s)^2 - 2 \cdot 3s \cdot 4s \cdot \cos 120^\circ$$
$$= s^2 \left(25 - 24\left(-\frac{1}{2}\right)\right) = 37s^2.$$

By symmetry, $\triangle A'B'C'$ is also equilateral and therefore is similar to

$\triangle ABC$ with similarity ratio $\sqrt{37}$. Hence the ratio of their areas is $37 : 1$.

OR