UNIT 3 EXERCISES 11-15

2D GEO WORD

2003A

11. (C) Rescaling to different units does not affect the ratio of the areas, so let the perimeter be 12. Each side of the square then has length 3, and each side of the triangle has length 4. The diameter of the circle circumscribing the square is the diagonal of the square, $3\sqrt{2}$. Thus $A = \pi(3\sqrt{2}/2)^2 = 9\pi/2$. The altitude of the triangle is $2\sqrt{3}$, so the radius of the circle circumscribing the triangle is $4\sqrt{3}/3$, and $B = \pi(4\sqrt{3}/3)^2 = 16\pi/3$. Therefore

$$\frac{A}{B} = \frac{9\pi}{2}\frac{3}{16\pi} = \frac{27}{32}.$$

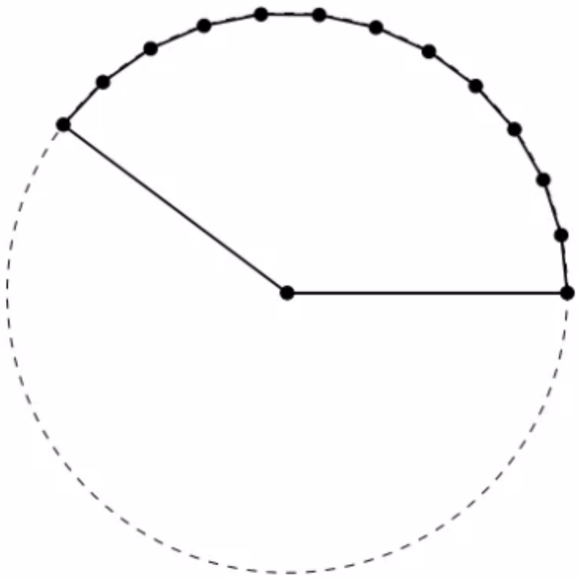
2015A

11. **Answer (D):** If the smaller circle is in the interior of the larger circle, there are no common tangent lines. If the smaller circle is internally tangent to the larger circle, there is exactly one common tangent line. If the circles intersect at two points, there are exactly two common tangent lines. If the circles are externally tangent, there are exactly three tangent lines. Finally, if the circles do not intersect, there are exactly four tangent lines. Therefore, k can be any of the numbers 0, 1, 2, 3, or 4, which gives 5 possibilities.

2017A 11. **Answer (D):** If the polygon has n sides and the degree measure of the forgotten angle is α, then $(n-2)180 = 2017 + \alpha$. Because $0 < \alpha < 180$,

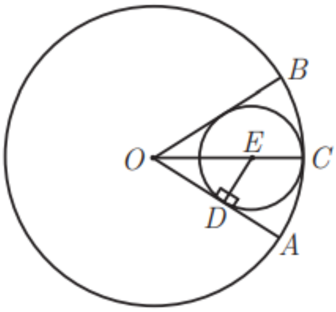
$$2017 < (n-2)180 < 2197,$$

which implies that $n = 14$, the angle sum is 2160, and $\alpha = 143$. To see that such a polygon exists, draw a circle and a central angle of measure 143°, and divide the minor arc spanned by the angle into 12 small arcs. The polygon is then formed by the two radii and 12 small chords, as illustrated.



2008A

13. **Answer (B):** Let r and R be the radii of the smaller and larger circles, respectively. Let E be the center of the smaller circle, let $\overline{OC}$ be the radius of the larger circle that contains E, and let D be the point of tangency of the smaller circle to $\overline{OA}$. Then $OE = R - r$, and because $\triangle EDO$ is a $30-60-90^\circ$ triangle, $OE = 2DE = 2r$. Thus $2r = R - r$, so $\frac{r}{R} = \frac{1}{3}$. The ratio of the areas is $(\frac{1}{3})^2 = \frac{1}{9}$.



2015B

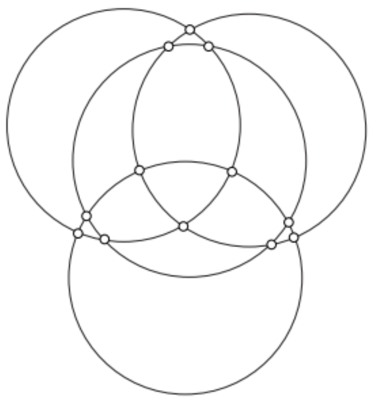
13. **Answer (B):** Because $\angle BAC$ and $\angle BDC$ intercept the same arc, $\angle BDC = 70^\circ$. Then $\angle ADC = 110^\circ$ and $\angle ABC = 180^\circ - \angle ADC = 70^\circ$. Thus $\triangle ABC$ is isosceles, and therefore $AC = BC = 6$.

2001

14. **(D)** Each of the $\binom{9}{2} \equiv 9C2 = 36$ pairs of vertices determines two equilateral triangles, for a total of 72 triangles. However, the three triangles $A_1A_4A_7$, $A_2A_5A_8$, and $A_3A_6A_9$ are each counted 3 times, resulting in an overcount of 6. Thus, there are 66 distinct equilateral triangles.

2002B

14. **(D)** Each pair of circles has at most two intersection points. There are $\binom{4}{2} = 6$ pairs of circles, so there are at most $6 \times 2 = 12$ points of intersection. The following configuration shows that 12 points of intersection are indeed possible:



2015B 14. **Answer (D):** Let x equal the area of the circle, y the area of the triangle, and z the area of the overlapped sector. The answer is $(x - z) - (y - z) = x - y$. The area of the circle is 4π and the area of the triangle is $\frac{\sqrt{3}}{4} \cdot 4^2 = 4\sqrt{3}$, so the result is $4(\pi - \sqrt{3})$.

2008B 15. **Answer (C):** The region inside S but outside R consists of four triangles, each of which has two sides of length 1. The angle between those two sides is $360^\circ - 90^\circ - 4 \cdot 60^\circ = 30^\circ$. Thus the area of each triangle is

$$\frac{1}{2} \cdot 1 \cdot 1 \cdot \sin 30^\circ = \frac{1}{4},$$

so the required area is $4 \cdot \frac{1}{4} = 1$.

2016A 15. **Answer (D):** Let X be the foot of the perpendicular from P to $\overline{QQ'}$, and let Y be the foot of the perpendicular from Q to $\overline{RR'}$. By the Pythagorean Theorem,

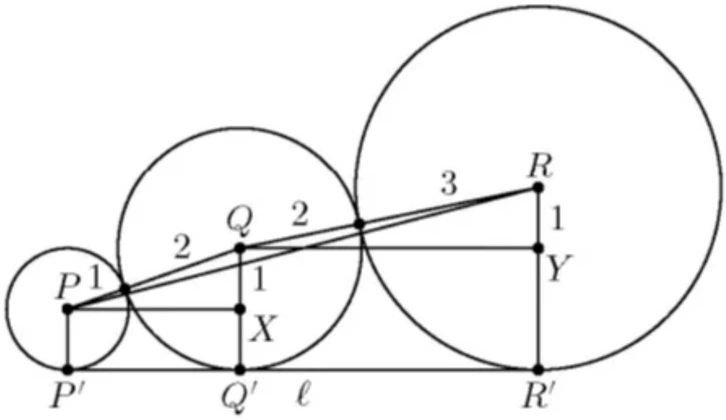
$$P'Q' = PX = \sqrt{(2+1)^2 - (2-1)^2} = \sqrt{8}$$

and

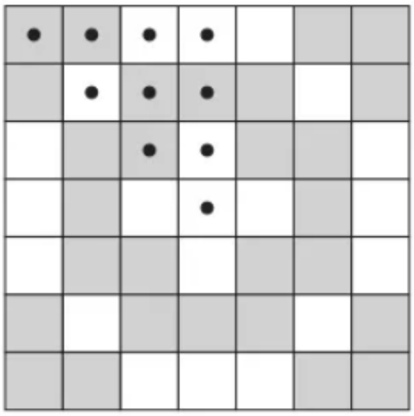
$$Q'R' = QY = \sqrt{(3+2)^2 - (3-2)^2} = \sqrt{24}.$$

The required area can be computed as the sum of the areas of the two smaller trapezoids, $PQQ'P'$ and $QRR'Q'$, minus the area of the large trapezoid, $PRR'P'$:

$$\frac{1+2}{2}\sqrt{8} + \frac{2+3}{2}\sqrt{24} - \frac{1+3}{2}\left(\sqrt{8}+\sqrt{24}\right) = \sqrt{6} - \sqrt{2}.$$



2018A 15. **Answer (B):** None of the squares that are marked with dots in the sample scanning code shown below can be mapped to any other marked square by reflections or non-identity rotations. Therefore these 10 squares can be arbitrarily colored black or white in a symmetric scanning code, with the exception of "all black" and "all white". On the other hand, reflections or rotations will map these squares to all the other squares in the scanning code, so once these 10 colors are specified, the symmetric scanning code is completely determined. Thus there are $2^{10} - 2 = 1022$ symmetric scanning codes.



OR

The diagram below shows the orbits of each square under rotations and reflections. Because the scanning code must look the same under these transformations, all squares in the same orbit must get the same color, but one is free to choose the color for each orbit, except for the choice of "all black" and "all white". Because there are 10 orbits, there are $2^{10} - 2 = 1022$ symmetric scanning codes.

A	*B*	*C*	*D*	*C*	*B*	*A*
B	*E*	*F*	*G*	*F*	*E*	*B*
C	*F*	*H*	*I*	*H*	*F*	*C*
D	*G*	*I*	*J*	*I*	*G*	*D*
C	*F*	*H*	*I*	*H*	*F*	*C*
B	*E*	*F*	*G*	*F*	*E*	*B*
A	*B*	*C*	*D*	*C*	*B*	*A*