UNIT 21 EXERCISES 11-15

SERIES

2009A

11. **Answer (E):** The outside square for F_n has 4 more diamonds on its boundary than the outside square for F_{n-1}. Because the outside square of F_2 has 4 diamonds, the outside square of F_n has $4(n-2)+4=4(n-1)$ diamonds. Hence the number of diamonds in figure F_n is the number of diamonds in F_{n-1} plus $4(n-1)$, or

$$\begin{aligned}&1+4+8+12+\cdots+4(n-2)+4(n-1)\\=&1+4(1+2+3+\cdots+(n-2)+(n-1))\\=&1+4\frac{(n-1)n}{2}\\=&1+2(n-1)n.\end{aligned}$$

Therefore figure F_{20} has $1+2\cdot 19\cdot 20=761$ diamonds.

2009B

11. **Answer (D):** On Monday, day 1, the birds find $\frac{1}{4}$ quart of millet in the feeder. On Tuesday they find

$$\frac{1}{4}+\frac{3}{4}\cdot\frac{1}{4}$$

quarts of millet. On Wednesday, day 3, they find

$$\frac{1}{4}+\frac{3}{4}\cdot\frac{1}{4}+\left(\frac{3}{4}\right)^2\cdot\frac{1}{4}$$

quarts of millet. The number of quarts of millet they find on day n is

$$\frac{1}{4}+\frac{3}{4}\cdot\frac{1}{4}+\left(\frac{3}{4}\right)^2\cdot\frac{1}{4}+\cdots+\left(\frac{3}{4}\right)^{n-1}\cdot\frac{1}{4}=\frac{(\frac{1}{4})(1-(\frac{3}{4})^n)}{1-\frac{3}{4}}=1-\left(\frac{3}{4}\right)^n.$$

The birds always find $\frac{3}{4}$ quart of other seeds, so more than half the seeds are millet if $1-(\frac{3}{4})^n>\frac{3}{4}$, that is, when $(\frac{3}{4})^n<\frac{1}{4}$. Because $(\frac{3}{4})^4=\frac{81}{256}>\frac{1}{4}$ and $(\frac{3}{4})^5=\frac{243}{1024}<\frac{1}{4}$, this will first occur on day 5 which is Friday.

2004B

12. **(C)** Let a_k be the k^{th} term of the sequence. For $k\ge 3$,

$$a_{k+1}=a_{k-2}+a_{k-1}-a_k,\quad \text{so}\quad a_{k+1}-a_{k-1}=-(a_k-a_{k-2}).$$

Because the sequence begins

$$2001, 2002, 2003, 2000, 2005, 1998, \ldots,$$

it follows that the odd-numbered terms and the even-numbered terms each form arithmetic progressions with common differences of 2 and -2, respectively. The 2004^{th} term of the original sequence is the 1002^{nd} term of the sequence 2002, 2000, 1998,..., and that term is $2002+1001(-2)=0$.

2008B

12. **Answer (B):** Because the mean of the first n terms is n, their sum is n^2. Therefore the nth term is $n^2-(n-1)^2=2n-1$, and the 2008th term is $2\cdot 2008-1=4015$.

2009B

12. **Answer (E):** Let the nth term of the series be ar^{n-1}. Because

$$\frac{8!}{7!} = \frac{ar^7}{ar^4} = r^3 = 8,$$

it follows that $r = 2$ and the first term is $a = \frac{7!}{r^4} = \frac{7!}{16} = 315$.

1999

13. **(C)** Since $a_{n+1} = \sqrt[3]{99} \cdot a_n$ for all $n \geq 1$, it follows that $a_1, a_2, a_3, \ldots$ is a geometric sequence whose first term is 1 and whose common ratio is $r = \sqrt[3]{99}$. Thus

$$a_{100} = a_1 \cdot r^{100-1} = \left(\sqrt[3]{99}\right)^{99} = 99^{33}.$$

2002B

13. **(B)** Let $n, n+1, \ldots, n+17$ be the 18 consecutive integers. Then the sum is

$$18n + (1 + 2 + \cdots 17) = 18n + \frac{17 \cdot 18}{2} = 9(2n + 17).$$

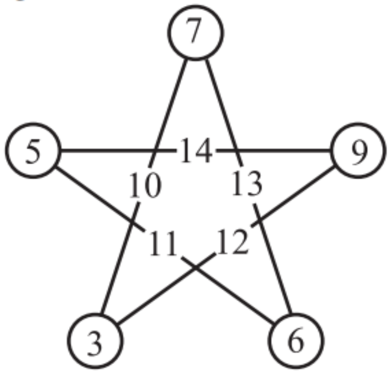
Since 9 is a perfect square, $2n + 17$ must also be a perfect square. The smallest value of n for which this occurs is $n = 4$, so $9(2n + 17) = 9 \cdot 25 = 225$.

2005A 13. **(D)** Each number appears in two sums, so the sum of the sequence is

$$2(3+5+6+7+9)=60.$$

The middle term of a five-term arithmetic sequence is the mean of its terms, so $60/5=12$ is the middle term.

The figure shows an arrangement of the five numbers that meets the requirement.



2004A 14. **(A)** The terms of the arithmetic progression are 9, $9+d$, and $9+2d$ for some real number d. The terms of the geometric progression are 9, $11+d$, and $29+2d$. Therefore

$$(11+d)^2=9(29+2d) \quad \text{so} \quad d^2+4d-140=0.$$

Thus $d=10$ or $d=-14$. The corresponding geometric progressions are 9, 21, 49 and 9, -3, 1, so the smallest possible value for the third term of the geometric progression is 1.

2013A 14. **Answer (B):**

Because the terms form an arithmetic sequence,

$$\log_{12} y = \frac{1}{2}(\log_{12} 162 + \log_{12} 1250) = \frac{1}{2}\log_{12}(162 \cdot 1250)$$
$$= \frac{1}{2}\log_{12}(2^2 3^4 5^4) = \log_{12}(2 \cdot 3^2 5^2).$$

Then

$$\log_{12} x = \frac{1}{2}(\log_{12} 162 + \log_{12} y) = \frac{1}{2}\left(\log_{12}(2 \cdot 3^4) + \log_{12}(2 \cdot 3^2 5^2)\right)$$
$$= \frac{1}{2}\log_{12}(2^2 3^6 5^2) = \log_{12}(2 \cdot 3^3 5) = \log_{12} 270.$$

Therefore $x = 270$.

OR

If $(B_k) = (\log_{12} A_k)$ is an arithmetic sequence with common difference d, then (A_k) is a geometric sequence with common ratio $r = 12^d$. Therefore $162, x, y, z, 1250$ is a geometric sequence. Let r be their common ratio. Then $1250 = 162r^4$ and $r = \frac{5}{3}$. Thus $x = 162r = 162 \cdot \frac{5}{3} = 270$.

2014A 14. **Answer (C):** Let $d = b - a$ be the common difference of the arithmetic progression. Then $b = a + d$, $c = a + 2d$, and because a, c, b is a geometric progression,

$$\frac{a+2d}{a} = \frac{a+d}{a+2d}.$$

Thus $(a + 2d)(a + 2d) = a(a + d)$, which simplifies to $3ad + 4d^2 = 0$. Because $d > 0$, it follows that $3a + 4d = 0$ and therefore $a = -4k$ and $d = 3k$ for some positive integer k. Thus $c = (-4k) + 2(3k) = 2k$, and the smallest value of c is $2 \cdot 1 = 2$.

2016B 14. **Answer (E):** Let r be the common ratio of the geometric series; then

$$S = \frac{1}{r} + 1 + r + r^2 + \cdots = \frac{\frac{1}{r}}{1-r} = \frac{1}{r-r^2}.$$

Because $S > 0$, the smallest value of S occurs when the value of $r - r^2$ is maximized. The graph of $f(r) = r - r^2$ is a downward-opening parabola with vertex $(\frac{1}{2}, \frac{1}{4})$, so the smallest possible value of S is $\frac{1}{(\frac{1}{4})} = 4$. The optimal series is $2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$.

2007B 15. **Answer (E):** The terms involving odd powers of r form the geometric series $ar + ar^3 + ar^5 + \cdots$. Thus

$$7 = a + ar + ar^2 + \cdots = \frac{a}{1-r},$$

and

$$3 = ar + ar^3 + ar^5 + \cdots = \frac{ar}{1-r^2} = \frac{a}{1-r} \cdot \frac{r}{1+r} = \frac{7r}{1+r}.$$

Therefore $r = 3/4$. It follows that $a/(1/4) = 7$, so $a = 7/4$ and

$$a + r = \frac{7}{4} + \frac{3}{4} = \frac{5}{2}.$$

OR

The sum of the terms involving even powers of r is $7 - 3 = 4$. Therefore

$$3 = ar + ar^3 + ar^5 + \cdots = r(a + ar^2 + ar^4 + \cdots) = 4r,$$

so $r = 3/4$. As in the first solution, $a = 7/4$ and $a + r = 5/2$.

2014B 15. **Answer (C):** Because $k \ln k = \ln(k^k)$ and the log of a product is the sum of the logs, $p = \ln \prod_{k=1}^{6} k^k$. Therefore e^p is the integer $1^1 \cdot 2^2 \cdot 3^3 \cdot 4^4 \cdot 5^5 \cdot 6^6 = 2^{16} \cdot 3^9 \cdot 5^5$, and the largest power of 2 dividing e^p is 2^{16}.