UNIT 2 EXERCISES 11-15

3D GEO

2008A 11. **Answer (C):** The sum of the six numbers on each cube is $1+2+4+8+16+32 = 63$. The three pairs of opposite faces have numbers with sums $1 + 32 = 33$, $2 + 16 = 18$, and $4 + 8 = 12$. On the two lower cubes, the numbers on the four visible faces have the greatest sum when the 4 and the 8 are hidden. On the top cube, the numbers on the five visible faces have the greatest sum when the 1 is hidden. Thus the greatest possible sum is $3 \cdot 63 - 2 \cdot (4 + 8) - 1 = 164$.

2008B 11. **Answer (A):** The portion of the mountain that is above the water forms a cone that is similar to the entire mountain. The ratio of the volumes of the cones is the cube of the ratio of their heights. Let d be the depth of the ocean, in feet. Then the height of the mountain above the water is $8000 - d$ feet, and

$$\frac{(8000-d)^3}{8000^3} = \frac{1}{8}.$$

Taking cube roots on both sides gives

$$\frac{8000-d}{8000} = \frac{1}{2},$$

from which $16{,}000 - 2d = 8000$, and $d = 4000$.

2014A 11. **Answer (C):** Let d be the remaining distance after one hour of driving, and let t be the remaining time until his flight. Then $d = 35(t+1)$, and $d = 50(t-0.5)$. Solving gives $t = 4$ and $d = 175$. The total distance from home to the airport is $175 + 35 = 210$ miles.

OR

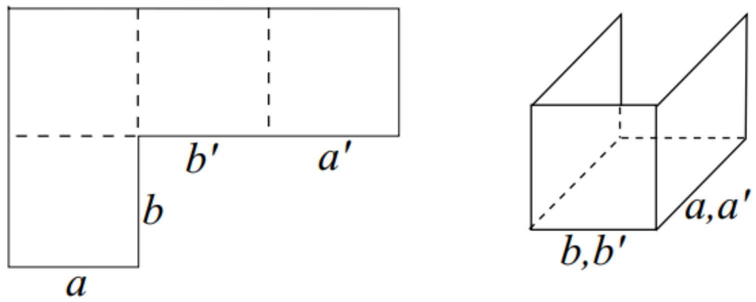
Let d be the distance between David's home and the airport. The time required to drive the entire distance at 35 MPH is $\frac{d}{35}$ hours. The time required to drive at 35 MPH for the first 35 miles and 50 MPH for the remaining $d - 35$ miles is $1 + \frac{d-35}{50}$. The second trip is 1.5 hours quicker than the first, so

$$\frac{d}{35} - \left(1 + \frac{d-35}{50}\right) = 1.5.$$

Solving yields $d = 210$ miles.

2003A

13. (E) If the polygon is folded before the fifth square is attached, then edges a and a' must be joined, as must b and b'. The fifth face of the cube can be attached at any of the six remaining edges.



2003B

13. (B) Let r be the radius of the sphere and cone, and let h be the height of the cone. Then the conditions of the problem imply that

$$\frac{3}{4}\left(\frac{4}{3}\pi r^3\right) = \frac{1}{3}\pi r^2 h, \quad \text{so } h = 3r.$$

Therefore, the ratio of h to r is $3 : 1$.

2014B

14. **Answer (D):** Denote the edge lengths by x, y, and z. The surface area is $2(xy+yz+zx) = 94$ and the sum of the lengths of the edges is $4(x+y+z) = 48$. Therefore $144 = (x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx) = x^2+y^2+z^2+94$, so $x^2+y^2+z^2 = 50$. By the Pythagorean Theorem applied twice, each of the 4 internal diagonals has length $\sqrt{50}$, and their total length is $4\sqrt{50} = 20\sqrt{2}$. A right rectangular prism with edge lengths 3, 4, and 5 satisfies the conditions of the problem.

2016A

14. **Answer (C):** The sum of the four numbers on the vertices of each face must be $\frac{1}{6}\cdot 3\cdot(1+2+\cdots+8) = 18$. The only sets of four of the numbers that include 1 and have a sum of 18 are $\{1,2,7,8\}$, $\{1,3,6,8\}$, $\{1,4,5,8\}$, and $\{1,4,6,7\}$. Three of these sets contain both 1 and 8. Because two specific vertices can belong to at most two faces, the vertices of one face must be labeled with the numbers $1,4,6,7$, and two of the faces must include vertices labeled 1 and 8. Thus 1 and 8 must mark two adjacent vertices. The cube can be rotated so that the vertex labeled 1 is at the lower left front, and the vertex labeled 8 is at the lower right front. The numbers 4, 6, and 7 must label vertices on the left face. There are $3! = 6$ ways to assign these three labels to the three remaining vertices of the left face. Then the numbers 5, 3, and 2 must label the vertices of the right face adjacent to the vertices labeled 4, 6, and 7, respectively. Hence there are 6 possible arrangements.

2017B

14. **Answer (E):** A frustum is constructed by removing a right circular cone from a larger right circular cone. The volume of the given frustum is the volume of a right circular cone with a 4-inch-diameter base and a height of 8 inches, minus the volume of a right circular cone with a 2-inch-diameter base and a height of 4 inches. (The stated heights come from considering similar right triangles.) Because the volume of a right circular cone is $\frac{1}{3}\pi r^2 h$, the volume of the frustum is

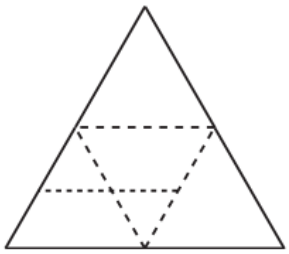
$$\frac{1}{3}\pi \cdot 2^2 \cdot 8 - \frac{1}{3}\pi \cdot 1^2 \cdot 4 = \frac{28}{3}\pi.$$

The volume of the top cone of the novelty is $\frac{1}{3}\pi \cdot 2^2 \cdot 4 = \frac{16}{3}\pi$. The requested volume of ice cream is the sum of the volume of each part of the novelty, namely $\frac{28}{3}\pi + \frac{16}{3}\pi = \frac{44}{3}\pi$.

Note: In general, the volume of a frustum with height h and base radii R and r is $\frac{1}{3}\pi h(r^2 + rR + R^2)$.

2001

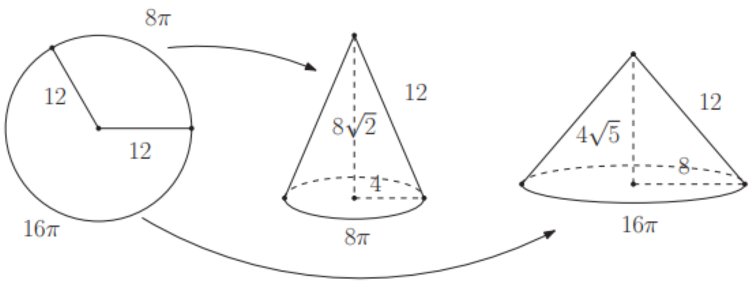
15. **(B)** Unfold the tetrahedron onto a plane. The two opposite-edge midpoints become the midpoints of opposite sides of a rhombus with sides of length 1, so are now 1 unit apart. Folding back to a tetrahedron does not change the distance and it remains minimal.



2012B

15. **Answer (C):** Each sector forms a cone with slant height 12. The circumference of the base of the smaller cone is $\frac{120}{360} \cdot 2 \cdot 12 \cdot \pi = 8\pi$. Hence the radius of the base of the smaller cone is 4 and its height is $\sqrt{12^2 - 4^2} = 8\sqrt{2}$. Similarly, the circumference of the base of the larger cone is 16π. Hence the radius of the base of the larger cone is 8 and its height is $4\sqrt{5}$. The ratio of the volume of the smaller cone to the volume of larger cone is

$$\frac{\frac{1}{3}\pi \cdot 4^2 \cdot 8\sqrt{2}}{\frac{1}{3}\pi \cdot 8^2 \cdot 4\sqrt{5}} = \frac{\sqrt{10}}{10}.$$



2016B

15. **Answer (D):** Suppose that one pair of opposite faces of the cube are assigned the numbers a and b, a second pair of opposite faces are assigned the numbers c and d, and the remaining pair of opposite faces are assigned the numbers e and f. Then the needed sum of products is $ace + acf + ade + adf + bce + bcf + bde + bdf = (a+b)(c+d)(e+f)$. The sum of these three factors is $2+3+4+5+6+7 = 27$. A product of positive numbers whose sum is fixed is maximized when the factors are all equal. Thus the greatest possible value occurs when $a+b = c+d = e+f = 9$, as in $(a,b,c,d,e,f) = (2,7,3,6,4,5)$. This results in the value $9^3 = 729$.