UNIT 18 EXERCISES 11-15

ALGEBRA WORD PROBLEMS

2005A 11. **(E)** The first and last digits must be both odd or both even for their average to be an integer. There are $5 \cdot 5 = 25$ odd-odd combinations for the first and last digits. There are $4 \cdot 5 = 20$ even-even combinations that do not use zero as the first digit. Hence the total is 45.

2007A 11. **Answer (D):** A given digit appears as the hundreds digit, the tens digit, and the units digit of a term the same number of times. Let k be the sum of the units digits in all the terms. Then $S = 111k = 3 \cdot 37k$, so S must be divisible by 37. To see that S need not be divisible by any larger prime, note that the sequence 123, 231, 312 gives $S = 666 = 2 \cdot 3^2 \cdot 37$.

2017B 11. **Answer (B):** The monotonous positive integers with one digit or increasing digits can be put into a one-to-one correspondence with the nonempty subsets of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The number of such subsets is $2^9 - 1 = 511$. The monotonous positive integers with one digit or decreasing digits can be put into a one-to-one correspondence with the subsets of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ other than $\emptyset$ and $\{0\}$. The number of these is $2^{10} - 2 = 1022$. The single-digit numbers are included in both sets, so there are $511 + 1022 - 9 = 1524$ monotonous positive integers.

2003A 12. (E) Let R1, ..., R5 and B3, ..., B6 denote the numbers on the red and blue cards, respectively. Note that R4 and R5 divide evenly into only B4 and B5, respectively. Thus the stack must be R4, B4, ..., B5, R5, or the reverse. Since R2 divides evenly into only B4 and B6, we must have R4, B4, R2, B6, ..., B5, R5, or the reverse. Since R3 divides evenly into only B3 and B6, the stack must be R4, B4, R2, B6, R3, B3, R1, B5, R5, or the reverse. In either case, the sum of the middle three cards is 12.

2009A

12. **Answer (B):** The only such number is 54. A single-digit number would have to satisfy $6u = u$, implying $u = 0$, which is impossible. A two-digit number would have to satisfy $10t + u = 6(t + u)$, so $4t = 5u$ and then necessarily $t = 5$ and $u = 4$; hence the number is 54. A three-digit number would have to satisfy $100h + 10t + u = 6(h + t + u)$ or $94h + 4t = 5u$. But the left side of the expression is at least 94 while the right side of the expression is at most 45, so no solution is possible.

2011B

13. **Answer (B):** The largest pairwise difference is 9, so $w - z = 9$. Let n be either x or y. Because n is between w and z,

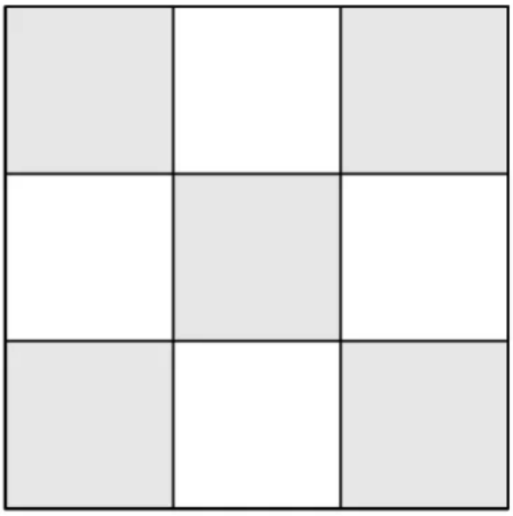
$$9 = w - z = (w - n) + (n - z).$$

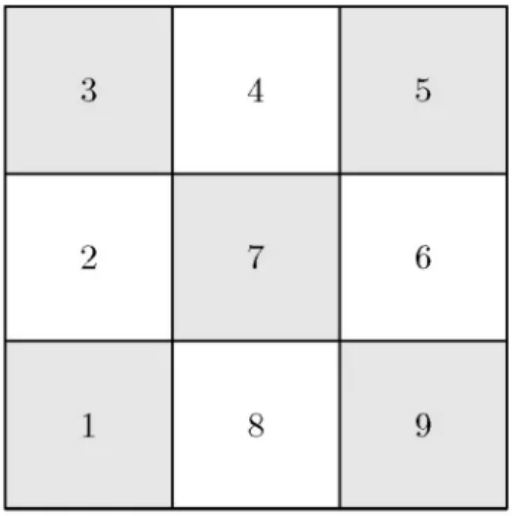
Therefore the positive differences $w - n$ and $n - z$ must sum to 9. The given pairwise differences that sum to 9 are $3 + 6$ and $4 + 5$. The remaining pairwise difference must be $x - y = 1$.

The second largest pairwise difference is 6, so either $w - y = 6$ or $x - z = 6$. In the first case the set of four numbers may be expressed as $\{w, w-5, w-6, w-9\}$. Hence $4w - 20 = 44$, so $w = 16$. In the second case $w - x = 3$, and the four numbers may be expressed as $\{w, w - 3, w - 4, w - 9\}$. Therefore $4w - 16 = 44$, so $w = 15$. The sum of the possible values for w is $16 + 15 = 31$.

Note: The possible sets of four numbers are $\{16, 11, 10, 7\}$ and $\{15, 12, 11, 6\}$.

2016B 12. **Answer (C):** Shade the squares in a checkerboard pattern as shown in the first figure. Because consecutive numbers must be in adjacent squares, the shaded squares will contain either five odd numbers or five even numbers. Because there are only four even numbers available, the shaded squares contain the five odd numbers. Thus the sum of the numbers in all five shaded squares is $1+3+5+7+9=25$. Because all but the center add up to $18=25-7$, the center number must be 7. The situation described is actually possible, as the second figure demonstrates.





2018A

12. **Answer (C):** If $1 \in S$, then S can have only 1 element, not 6 elements. If 2 is the least element of S, then 2, 3, 5, 7, 9, and 11 are available to be in S, but 3 and 9 cannot both be in S, so the largest possible size of S is 5. If 3 is the least element, then 3, 4, 5, 7, 8, 10, and 11 are available, but at most one of 4 and 8 can be in S and at most one of 5 and 10 can be in S, so again S has size at most 5. The set $S = \{4, 6, 7, 9, 10, 11\}$ has the required property, so 4 is the least possible element of S.

OR

At most one integer can be selected for S from each of the following 6 groups: $\{1, 2, 4, 8\}$, $\{3, 6, 12\}$, $\{5, 10\}$, $\{7\}$, $\{9\}$, and $\{11\}$. Because S consists of 6 distinct integers, exactly one integer must be selected from each of these 6 groups. Therefore 7, 9, and 11 must be in S. Because 9 is in S, 3 must not be in S. This implies that either 6 or 12 must be selected from the second group, so neither 1 nor 2 can be selected from the first group. If 4 is selected from the first group, the collection of integers $\{4, 5, 6, 7, 9, 11\}$ is one possibility for the set S. Therefore 4 is the least possible element of S.

Note: The two collections given in the solutions are the only ones with least element 4 that have the given property. This problem is a special case of the following result of Paul Erdős from the 1930s: Given n integers $a_1, a_2, \ldots, a_n$, no one of them dividing any other, with $a_1 < a_2 < \cdots < a_n \leq 2n$, let the integer k be determined by the inequalities $3^k < 2n < 3^{k+1}$. Then $a_1 \geq 2^k$, and this bound is sharp.

2018A

13. **Answer (D):** Let S be the set of integers, both negative and nonnegative, having the given form. Increasing the value of a_i by 1 for $0 \le i \le 7$ creates a one-to-one correspondence between S and the ternary (base 3) representation of the integers from 0 through $3^8 - 1$, so S contains $3^8 = 6561$ elements. One of those is 0, and by symmetry, half of the others are positive, so S contains $1 + \frac{1}{2} \cdot (6561 - 1) = 3281$ elements.

OR

First note that if an integer N can be written in this form, then $N - 1$ can also be written in this form as long as not all the a_i in the representation of N are equal to -1. A procedure to alter the representation of N so that it will represent $N - 1$ instead is to find the least value of i such that $a_i \ne -1$, reduce the value of that a_i by 1, and set $a_i = 1$ for all lower values of i. By the formula for the sum of a finite geometric series, the greatest integer that can be written in the given form is

$$\frac{3^8 - 1}{3 - 1} = 3280.$$

Therefore, 3281 nonnegative integers can be written in this form, namely all the integers from 0 through 3280, inclusive. (The negative integers from -3280 through -1 can also be written in this way.)

OR

Think of the indicated sum as an expansion in base 3 using "digits" -1, 0, and 1. Note that the leftmost digit a_k of any positive integer that can be written in this form cannot be negative and therefore must be 1. Then there are 3 choices for each of the remaining k digits to the right of a_k, resulting in 3^k positive integers that can be written in the indicated form. Thus there are

$$\sum_{k=0}^{7} 3^k = \frac{3^8 - 1}{3 - 1} = 3280$$

positive numbers of the indicated form. Because 0 can also be written in this form, the number of nonnegative integers that can be written in the indicated form is 3281.

2010B

14. **Answer (B):** Note that $3M > (a+b)+c+(d+e) = 2010$, so $M > 670$. Because M is an integer $M \geq 671$. The value of 671 is achieved if $(a,b,c,d,e) = (669,1,670,1,669)$.

2012B

14. **Answer (A):** The smallest initial number for which Bernardo wins after one round is the smallest integer solution of $2n + 50 \geq 1000$, which is 475. The smallest initial number for which he wins after two rounds is the smallest integer solution of $2n + 50 \geq 475$, which is 213. Similarly, the smallest initial numbers for which he wins after three and four rounds are 82 and 16, respectively. There is no initial number for which Bernardo wins after more than four rounds. Thus $N = 16$, and the sum of the digits of N is 7.

2002B

15. **(D)** Let a denote the leftmost digit of N and let x denote the three-digit number obtained by removing a. Then $N = 1000a + x = 9x$ and it follows that $1000a = 8x$. Dividing both sides by 8 yields $125a = x$. All the values of a in the range 1 to 7 result in three-digit numbers.

2005B

15. **(D)** The sum of the digits 1 through 9 is 45, so the sum of the eight digits is between 36 and 44, inclusive. The sum of the four units digits is between $1+2+3+4=10$ and $6+7+8+9=30$, inclusive, and also ends in 1. Therefore the sum of the units digits is either 11 or 21. If the sum of the units digits is 11, then the sum of the tens digits is 21, so the sum of all eight digits is 32, an impossibility. If the sum of the units digits is 21, then the sum of the tens digits is 20, so the sum of all eight digits is 41. Thus the missing digit is $45-41=4$. Note that the numbers 13, 25, 86, and 97 sum to 221.

OR

Each of the two-digit numbers leaves the same remainder when divided by 9 as does the sum of its digits. Therefore the sum of the four two-digit numbers leaves the same remainder when divided by 9 as the sum of all eight digits. Let d be the missing digit. Because 221 when divided by 9 leaves a remainder of 5, and the sum of the digits from 1 through 9 is 45, the number $(45-d)$ must leave a remainder of 5 when divided by 9. Thus $d=4$.

2015A

15. **Answer (C):** The numerator and denominator of this fraction have no common factors. To express the fraction as a decimal requires rewriting it with a power of 10 as the denominator. The smallest denominator that permits this is 10^{26}:

$$\frac{123\,456\,789}{2^{26}\cdot 5^4} = \frac{123\,456\,789\cdot 5^{22}}{2^{26}\cdot 5^4\cdot 5^{22}} = \frac{123\,456\,789\cdot 5^{22}}{10^{26}},$$

so the numeral will have 26 places after the decimal point. In fact

$$\frac{123\,456\,789}{2^{26}\cdot 5^4} = 0.00294\,34392\,21382\,14111\,32812\,5.$$

2018B 15. **Answer (A):** Let $\underline{a}\,\underline{b}\,\underline{c}$ be a 3-digit positive odd multiple of 3 that does not include the digit 3. There are 8 possible values for a, namely 1, 2, 4, 5, 6, 7, 8, and 9, and 4 possible values for c, namely 1, 5, 7, and 9. The possible values of b can be put into three groups of the same size: $\{0,6,9\}$, $\{1,4,7\}$, and $\{2,5,8\}$. Recall that an integer is divisible by 3 if and only if the sum of its digits is divisible by 3. Thus for every possible pair of digits (a,c), the choices for b such that $\underline{a}\,\underline{b}\,\underline{c}$ is divisible by 3 constitute one of those groups. Hence the answer is $8 \cdot 4 \cdot 3 = 96$.

OR

There are $\frac{1}{2} \cdot \frac{1}{3} \cdot 900 = 150$ odd 3-digit multiples of 3. Those including the digit 3 have the form $\underline{a}\,\underline{b}\,\underline{3}$, $\underline{a}\,\underline{3}\,\underline{b}$, or $\underline{3}\,\underline{a}\,\underline{b}$. There are 30 of the first type, where the number $\underline{a}\,\underline{b}$ is one of 12, 15, 18, $\ldots$, 99. There are 15 of the second type, where the number $\underline{a}\,\underline{b}$ is one of 15, 21, 27, $\ldots$, 99. There are 17 of the third type, where the number $\underline{a}\,\underline{b}$ is one of 03, 09, 15, $\ldots$, 99. The numbers 303, 339, 363, 393, 633, and 933 are each counted twice, and 333 is counted 3 times. By the Inclusion–Exclusion Principle there are $150 - (30 + 15 + 17) + (1 \cdot 6 + 2 \cdot 1) = 96$ such numbers.