UNIT 15 EXERCISES 11-15

ALGEBRA

1999

11. **(A)** The locker labeling requires $137.94/0.02 = 6897$ digits. Lockers 1 through 9 require 9 digits, lockers 10 through 99 require $2 \cdot 90 = 180$ digits, and lockers 100 through 999 require $3 \cdot 900 = 2700$ digits. Hence the remaining lockers require $6897 - 2700 - 180 - 9 = 4008$ digits, so there must be $4008/4 = 1002$ more lockers, each using four digits. In all, there are $1002 + 999 = 2001$ student lockers.

2007B 11. **Answer (D):** Let x be the degree measure of $\angle A$. Then the degree measures of angles B, C, and D are $x/2$, $x/3$, and $x/4$, respectively. The degree measures of the four angles have a sum of 360, so

$$360 = x + \frac{x}{2} + \frac{x}{3} + \frac{x}{4} = \frac{25x}{12}.$$

Thus $x = (12 \cdot 360)/25 = 172.8 \approx 173$.

2010A 11. **Answer (C):** If $x = \log_b 7^7$, then $b^x = 7^7$. Thus

$$(7b)^x = 7^x \cdot b^x = 7^{x+7} = 8^x.$$

Because $x > 0$, it follows that $7b = 8$ and so $b = \frac{8}{7}$.

OR

Taking the logarithm of both sides gives us $(x+7)\log 7 = x \log 8$. Solving, we have $\frac{x+7}{x} = \frac{\log 8}{\log 7}$, $x \log 8 = x \log 7 + 7 \log 7$, $x(\log 8 - \log 7) = 7 \log 7$, and we have $x = \dfrac{\log 7^7}{\log \frac{8}{7}}$. Using the change of base rule for logarithms, $b = \frac{8}{7}$.

2012B

11. **Answer (C):** First assume $B = A - 1$. By the definition of number bases,

$$A^2 + 3A + 2 + 4(A-1) + 3 = 6(A + A - 1) + 9.$$

Simplifying yields $A^2 - 5A - 2 = 0$, which has no integer solutions.

Next assume $B = A + 1$. In this case

$$A^2 + 3A + 2 + 4(A+1) + 3 = 6(A + A + 1) + 9,$$

which simplifies to $A^2 - 5A - 6 = (A-6)(A+1) = 0$. The only positive solution is $A = 6$. Letting $A = 6$ and $B = 7$ in the original equation produces $132_6 + 43_7 = 69_{13}$, or $56 + 31 = 87$, which is true. The required sum is $A + B = 13$.

1999

12. **(C)** The x-coordinates of the intersection points are precisely the zeros of the polynomial $p(x) - q(x)$. This polynomial has degree at most three, so it has at most three zeros. Hence, the graphs of the fourth degree polynomial functions intersect at most three times. Finding an example to show that three intersection points can be achieved is left to the reader.

2001

12. **(B)** For integers not exceeding 2001, there are $\lfloor 2001/3 \rfloor = 667$ multiples of 3 and $\lfloor 2001/4 \rfloor = 500$ multiples of 4. The total, 1167, counts the $\lfloor 2001/12 \rfloor = 166$ multiples of 12 twice, so there are $1167 - 166 = 1001$ multiples of 3 or 4. From these we exclude the $\lfloor 2001/15 \rfloor = 133$ multiples of 15 and the $\lfloor 2001/20 \rfloor = 100$ multiples of 20, since these are multiples of 5. However, this excludes the $\lfloor 2001/60 \rfloor = 33$ multiples of 60 twice, so we must re-include these. The number of integers satisfying the conditions is $1001 - 133 - 100 + 33 = 801$.

2002B

12. **(D)** If $\frac{n}{20-n} = k^2$, for some $k \geq 0$, then $n = \frac{20k^2}{k^2+1}$. Since k^2 and k^2+1 have no common factors and n is an integer, k^2+1 must be a factor of 20. This occurs only when $k = 0, 1, 2,$ or 3. The corresponding values of n are 0, 10, 16, and 18.

2010B

12. **Answer (D):** Rewriting each logarithm in base 2 gives

$$\frac{\frac{1}{2}\log_2 x}{\frac{1}{2}} + \log_2 x + \frac{2\log_2 x}{2} + \frac{3\log_2 x}{3} + \frac{4\log_2 x}{4} = 40.$$

Therefore $5\log_2 x = 40$, so $\log_2 x = 8$, and $x = 256$.

OR

For $a \neq 0$ the expression $\log_{2^a}(x^a) = y$ if and only if $2^{ay} = x^a$. Thus $2^y = x$ and $y = \log_2 x$. Therefore the given equation is equivalent to $5\log_2 x = 40$, so $\log_2 x = 8$ and $x = 256$.

2002A

13. **(C)** A number x differs by one from its reciprocal if and only if $x - 1 = 1/x$ or $x + 1 = 1/x$. These equations are equivalent to $x^2 - x - 1 = 0$ and $x^2 + x - 1 = 0$. Solving these by the quadratic formula yields the positive solutions

$$\frac{1+\sqrt{5}}{2} \quad \text{and} \quad \frac{-1+\sqrt{5}}{2},$$

which are reciprocals of each other. The sum of the two numbers is $\sqrt{5}$.

2005B

13. **(D)** Since $4^{x_1} = 5$, $5^{x_2} = 6, \ldots, 127^{x_{124}} = 128$, we have

$$4^{7/2} = 128 = 127^{x_{124}} = (126^{x_{123}})^{x_{124}} = 126^{x_{123} \cdot x_{124}} = \cdots = 4^{x_1 x_2 \cdots x_{124}}.$$

So $x_1 x_2 \cdots x_{124} = 7/2$.

OR

We have

$$\begin{aligned} x_1 x_2 \cdots x_{124} &= \log_4 5 \cdot \log_5 6 \cdots \log_{127} 128 \\ &= \frac{\log 5}{\log 4} \cdot \frac{\log 6}{\log 5} \cdots \frac{\log 128}{\log 127} = \frac{\log 128}{\log 4} = \frac{\log 2^7}{\log 2^2} = \frac{7 \log 2}{2 \log 2} = \frac{7}{2}. \end{aligned}$$

2008B

14. **Answer (C):** The given information implies that $2\pi \log_{10}(a^2) = \log_{10}(b^4)$ or, equivalently, that $4\pi \log_{10} a = 4 \log_{10} b$. Thus

$$\log_a b = \frac{\log_{10} b}{\log_{10} a} = \pi.$$

2018A

14. **Answer (D):** By the change-of-base formula, the given equation is equivalent to

$$\begin{aligned}\frac{\log 4}{\log 3x} &= \frac{\log 8}{\log 2x}\\ \frac{2\log 2}{\log 3+\log x} &= \frac{3\log 2}{\log 2+\log x}\\ 2\log 2+2\log x &= 3\log 3+3\log x\\ \log x &= 2\log 2-3\log 3\\ \log x &= \log\frac{4}{27}.\end{aligned}$$

Therefore $x = \frac{4}{27}$, and the requested sum is $4 + 27 = 31$.

OR

Changing to base-2 logarithms transforms the given equation into

$$\begin{aligned}\frac{2}{\log_2 3x} &= \frac{3}{\log_2 2x}\\ 2\log_2 2x &= 3\log_2 3x\\ \log_2(2x)^2 &= \log_2(3x)^3\\ (2x)^2 &= (3x)^3,\end{aligned}$$

so $x = \frac{4}{27}$, and the requested sum is $4 + 27 = 31$.

2008A

15. **Answer (D):** The units digit of 2^n is 2, 4, 8, and 6 for $n = 1$, 2, 3, and 4, respectively. For $n > 4$, the units digit of 2^n is equal to that of 2^{n-4}. Thus for every positive integer j the units digit of 2^{4j} is 6, and hence 2^{2008} has a units digit of 6. The units digit of 2008^2 is 4. Therefore the units digit of k is 0, so the units digit of k^2 is also 0. Because 2008 is even, both 2008^2 and 2^{2008} are multiples of 4. Therefore k is a multiple of 4, so the units digit of 2^k is 6, and the units digit of $k^2 + 2^k$ is also 6.

2009A 15. **Answer (D):** Let k be a multiple of 4. For $k \geq 0$,

$$(k+1)\,i^{k+1} + (k+2)\,i^{k+2} + (k+3)\,i^{k+3} + (k+4)\,i^{k+4} = (k+1)\,i + (k+2)\,(-1) + (k+3)\,(-i) + (k+4) = 2 - 2i.$$

Thus when $n = 4 \cdot 24 = 96$, we have $i + 2i^2 + \cdots + ni^n = 24\,(2 - 2i) = 48 - 48i$. Adding the term $97i^{97} = 97i$ gives $(48 - 48i) + 97i = 48 + 49i$ when $n = 97$.

2009B 15. **Answer (B):**

Rearrange the equations to the form

$$x = \frac{\log(\frac{7}{3})}{\log\,(1 + f(r))}.$$

Because $f(r)$ is positive, for each answer choice, x will be largest when $f(r)$ is the smallest. Because $r > 0$, we have $\frac{r}{10} < r < 2r$. Because $r^2 < 9 < 10$ we have $\frac{r}{10} < \frac{1}{r}$. Finally, $\sqrt{r} < 10$, so $\frac{r}{10} < \sqrt{r}$.