UNIT 12 EXERCISES 11-15

PROBABILITY

2001

11. **(D)** Think of continuing the drawing until all five chips are removed form the box. There are ten possible orderings of the colors: RRRWW, RRWRW, RWRRW, WRRRW, RRWWR, RWRWR, WRRWR, RWWRR, WRWRR, and WWRRR. The six orderings that end in R represent drawings that would have ended when the second white chip was drawn.

OR

Imagine drawing until only one chip remains. If the remaining chip is red, then that draw would have ended when the second white chip was removed. The last chip will be red with probability $3/5$.

2005B 11. **(D)** There are

$$\binom{8}{2} = \frac{8!}{6! \cdot 2!} = 28$$

ways to choose the bills. A sum of at least \$20 is obtained by choosing both \$20 bills, one of the \$20 bills and one of the six smaller bills, or both \$10 bills. Hence the probability is

$$\frac{1 + 2 \cdot 6 + 1}{28} = \frac{14}{28} = \frac{1}{2}.$$

2010B 11. **Answer (E):** Each four-digit palindrome has digit representation *abba* with $1 \le a \le 9$ and $0 \le b \le 9$. The value of the palindrome is $1001a + 110b$. Because 1001 is divisible by 7 and 110 is not, the palindrome is divisible by 7 if and only if $b = 0$ or $b = 7$. Thus the requested probability is $\frac{2}{10} = \frac{1}{5}$.

2012A 11. **Answer (B):**

If Alex wins 3 rounds, Mel wins 2 rounds, and Chelsea wins 1 round, then the game's outcomes will be a permutation of AAAMMC, where the i^{th} letter represents the initial of the winner of the i^{th} round. There are

$$\frac{6!}{3!2!1!} = 60$$

such permutations.

Because each round has only one winner, it follows that $P(\text{M}) + P(\text{C}) = 1 - P(\text{A}) = \frac{1}{2}$. Also $P(\text{M}) = 2P(\text{C})$ and so $P(\text{M}) = \frac{1}{3}$ and $P(\text{C}) = \frac{1}{6}$.

The probability that Alex wins 3 rounds, Mel wins 2 rounds, and Chelsea wins 1 round is therefore

$$\frac{6!}{3!2!1!}\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)^2\left(\frac{1}{6}\right) = \frac{60}{2^3 \cdot 3^2 \cdot 6} = \frac{5}{36}.$$

2007A 12. **Answer (E):** The number $ad - bc$ is even if and only if ad and bc are both odd or are both even. Each of ad and bc is odd if both of its factors are odd, and even otherwise. Exactly half of the integers from 0 to 2007 are odd, so each of ad and bc is odd with probability $(1/2) \cdot (1/2) = 1/4$ and are even with probability $3/4$. Hence the probability that $ad - bc$ is even is

$$\frac{1}{4} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{3}{4} = \frac{5}{8}.$$

2007B

13. **Answer (D):** The light completes a cycle every 63 seconds. Leah sees the color change if and only if she begins to look within three seconds before the change from green to yellow, from yellow to red, or from red to green. Thus she sees the color change with probability $(3+3+3)/63 = 1/7$.

2016A

13. **Answer (A):** Let $N = 5k$, where k is a positive integer. There are $5k+1$ equally likely possible positions for the red ball in the line of balls. Number these $0, 1, 2, 3, \ldots, 5k-1, 5k$ from one end. The red ball will *not* divide the green balls so that at least $\frac{3}{5}$ of them are on the same side if it is in position $2k+1, 2k+2, \ldots, 3k-1$. This includes $(3k-1)-2k = k-1$ positions. The probability that $\frac{3}{5}$ or more of the green balls will be on the same side is therefore $1 - \frac{k-1}{5k+1} = \frac{4k+2}{5k+1}$.

Solving the inequality $\frac{4k+2}{5k+1} < \frac{321}{400}$ for k yields $k > \frac{479}{5} = 95\frac{4}{5}$. The value of k corresponding to the required least value of N is therefore 96, so $N = 480$. The sum of the digits of N is 12.

2005A 14. **(D)** A standard die has a total of 21 dots. For $1 \le n \le 6$, a dot is removed from the face with n dots with probability $n/21$. Thus the face that originally has n dots is left with an odd number of dots with probability $n/21$ if n is even and $1 - n/21$ if n is odd. Each face is the top face with probability $1/6$. Therefore the top face has an odd number of dots with probability

$$\frac{1}{6}\left(\left(1-\frac{1}{21}\right)+\frac{2}{21}+\left(1-\frac{3}{21}\right)+\frac{4}{21}+\left(1-\frac{5}{21}\right)+\frac{6}{21}\right)=\frac{1}{6}\left(3+\frac{3}{21}\right)$$
$$=\frac{1}{6}\cdot\frac{66}{21}=\frac{11}{21}.$$

OR

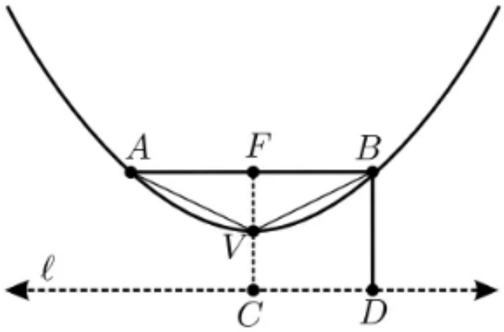
The probability that the top face is odd is $1/3$ if a dot is removed from an odd face, and the probability that the top face is odd is $2/3$ if a dot is removed from an even face. Because each dot has the probability $1/21$ of being removed, the top face is odd with probability

$$\left(\frac{1}{3}\right)\left(\frac{1+3+5}{21}\right)+\left(\frac{2}{3}\right)\left(\frac{2+4+6}{21}\right)=\frac{33}{63}=\frac{11}{21}.$$

2011A

14. **Answer (D):** Let ℓ be the directrix of the parabola, and let C and D be the projections of F and B onto ℓ, respectively. For any point in the parabola, its distance to F and to ℓ are the same. Because V and B are on the parabola, it follows that $p = FV = VC$ and $2p = FC = BD = FB$. By the Pythagorean Theorem, $VB = \sqrt{FV^2 + FB^2} = \sqrt{5}p$, and thus $\cos(\angle FVB) = \frac{FV}{VB} = \frac{p}{\sqrt{5}p} = \frac{\sqrt{5}}{5}$. Because $\angle AVB = 2(\angle FVB)$, it follows that

$$\cos(\angle AVB) = 2\cos^2(\angle FVB) - 1 = 2\left(\frac{\sqrt{5}}{5}\right)^2 - 1 = \frac{2}{5} - 1 = -\frac{3}{5}.$$



OR

Establish as in the first solution that $FV = p$, $FB = 2p$, and $VB = \sqrt{5}p$. Then $AB = 2 \cdot FB = 4p$, and by the Law of Cosines applied to $\triangle ABV$,

$$\cos\angle AVB = \frac{VA^2 + VB^2 - AB^2}{2(VA)(VB)} = \frac{5p^2 + 5p^2 - 16p^2}{2(5p^2)} = -\frac{3}{5}.$$

Note: The segment AB is called the *latus rectum*.

2010A

15. **Answer (D):** Let p be the requested probability. If the coin is flipped four times, the probability of heads and tails appearing twice is $\binom{4}{2}p^2(1-p)^2 = \frac{1}{6}$, and because $0 \le p \le 1$ it follows that $p(1-p) = \frac{1}{6}$. Solving for p yields $p = \frac{1}{6}(3 \pm \sqrt{3})$ and because $p < 1/2$, the answer is $p = \frac{1}{6}(3 - \sqrt{3})$.

2012A

15. **Answer (A):** There are $2^4 = 16$ possible initial colorings for the four corner squares. If their initial coloring is $BBBB$, one of the four cyclic permutations of $BBBW$, or one of the two cyclic permutations of $BWBW$, then all four corner squares are black at the end. If the initial coloring is $WWWW$, one of the four cyclic permutations of $BWWW$, or one of the four cyclic permutations of $BBWW$, then at least one corner square is white at the end. Hence all four corner squares are black at the end with probability $\frac{7}{16}$. Similarly, all four edge squares are black at the end with probability $\frac{7}{16}$. The center square is black at the end if and only if it was initially black, so it is black at the end with probability $\frac{1}{2}$. The probability that all nine squares are black at the end is $\frac{1}{2} \cdot \left(\frac{7}{16}\right)^2 = \frac{49}{512}$.

2015B 15. **Answer (D):** Rachelle needs a total of at least 14 points to get a 3.5 or higher GPA, so she needs a total of at least 6 points in English and History. The probability of a C in English is $1-\frac{1}{6}-\frac{1}{4}=\frac{7}{12}$, and the probability of a C in History is $1-\frac{1}{4}-\frac{1}{3}=\frac{5}{12}$. The probability that Rachelle earns exactly 6, 7, or 8 total points is computed as follows:

6 points: $\frac{1}{6}\cdot\frac{5}{12}+\frac{1}{4}\cdot\frac{1}{3}+\frac{7}{12}\cdot\frac{1}{4}=\frac{43}{144}$

7 points: $\frac{1}{6}\cdot\frac{1}{3}+\frac{1}{4}\cdot\frac{1}{4}=\frac{17}{144}$

8 points: $\frac{1}{6}\cdot\frac{1}{4}=\frac{6}{144}$

The probability that Rachelle will get at least a 3.5 GPA is

$$\frac{43}{144}+\frac{17}{144}+\frac{6}{144}=\frac{66}{144}=\frac{11}{24}.$$