UNIT 10 EXERCISES 11-15

COMBINATIONS

2012B

12. **Answer (E):** By symmetry, half of all such sequences end in zero. Of those, exactly one consists entirely of zeros. Each of the others contains a single subsequence of one or more consecutive ones beginning at position j and ending at position k with $1 \le j \le k \le 19$. Thus the number of sequences that meet the requirements is

$$2\left(1+\sum_{k=1}^{19}\sum_{j=1}^{k}1\right) = 2(1+(1+2+3+\cdots+19)) = 2\left(1+\frac{19\cdot 20}{2}\right) = 382.$$

OR

Let A be the set of zero-one sequences of length 20 where all the zeros appear together, and let B be the equivalent set of sequences where all the ones appear together. Set A contains one sequence with no zeros and 20 sequences with exactly one zero. Each sequence of A with more than one zero has a position where the first zero appears and a position where the last zero appears, so there are $\binom{20}{2} = 190$ such sequences, and thus $|A| = 1+20+190 = 211$. By symmetry $|B| = 211$. A sequence in $A \cap B$ begins with zero and contains from 1 to 20 zeros, or it begins with one and contains from 1 to 20 ones; thus $|A \cap B| = 40$. Therefore the required number of sequences equals

$$|A \cup B| = |A| + |B| - |A \cap B| = 211 + 211 - 40 = 382.$$

2014A 12. **Answer (D):**

Let the larger and smaller circles have radii R and r, respectively. Then the length of chord $\overline{AB}$ can be expressed as both r and $2R\sin 15^\circ$. The ratio of the areas of the circles is

$$\frac{\pi R^2}{\pi r^2} = \frac{1}{4\sin^2 15^\circ} = \frac{1}{2(1-\cos 30^\circ)} = \frac{1}{2-\sqrt{3}} = 2+\sqrt{3}.$$

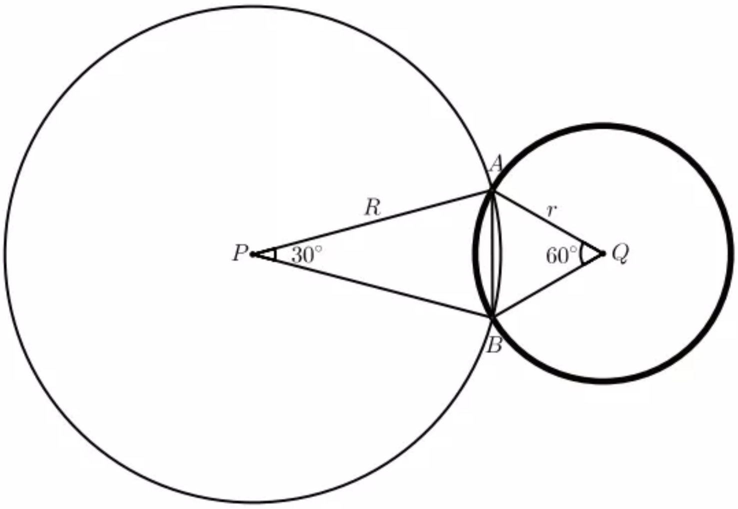
OR

Let the larger and smaller circles have radii R and r, and centers P and Q, respectively. Because $\triangle QAB$ is equilateral, it follows that $r = AB$. The Law of Cosines applied to $\triangle PBA$ gives

$$\begin{aligned} r^2 = AB^2 &= PA^2 + PB^2 - 2PA \cdot PB\cos 30^\circ \\ &= 2R^2 - 2R^2\cos 30^\circ = R^2(2-\sqrt{3}). \end{aligned}$$

Thus

$$\frac{\pi R^2}{\pi r^2} = \frac{1}{2-\sqrt{3}} = 2+\sqrt{3}.$$



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2017A 14. **Answer (C):** Let X be the set of ways to seat the five people in which Alice sits next to Bob. Let Y be the set of ways to seat the five people in which Alice sits next to Carla. Let Z be the set of ways to seat the five people in which Derek sits next to Eric. The required answer is $5! - |X \cup Y \cup Z|$. The Inclusion–Exclusion Principle gives

$$|X \cup Y \cup Z| = \big(|X|+|Y|+|Z|\big) - \big(|X \cap Y|+|X \cap Z|+|Y \cap Z|\big) + |X \cap Y \cap Z|.$$

Viewing Alice and Bob as a unit in which either can sit on the other's left side shows that there are $2 \cdot 4! = 48$ elements of X. Similarly there are 48 elements of Y and 48 elements of Z. Viewing Alice, Bob, and Carla as a unit with Alice in the middle shows that $|X \cap Y| = 2 \cdot 3! = 12$. Viewing Alice and Bob as a unit and Derek and Eric as a unit shows that $|X \cap Z| = 2 \cdot 2 \cdot 3! = 24$. Similarly $|Y \cap Z| = 24$. Finally, there are $2 \cdot 2 \cdot 2! = 8$ elements of $X \cap Y \cap Z$. Therefore $|X \cup Y \cup Z| = (48+48+48) - (12+24+24) + 8 = 92$, and the answer is $120 - 92 = 28$.

OR

There are three cases based on where Alice is seated.

• If Alice takes the first or last chair, then Derek or Eric must be seated next to her, Bob or Carla must then take the middle chair, and either of the remaining two individuals can be seated in either of the other two chairs. This gives a total of $2^4 = 16$ arrangements.

• If Alice is seated in the second or fourth chair, then Derek and Eric will take the seats on her two sides, and this can be done in two ways. Bob and Carla can be seated in the two remaining chairs in two ways, which yields a total of $2^3 = 8$ arrangements.

• If Alice sits in the middle chair, then Derek and Eric will be seated on her two sides, with Bob and Carla seated in the first and last chairs. This results in $2^2 = 4$ arrangements.

Thus there are $16 + 8 + 4 = 28$ possible arrangements in total.

2013A 15. **Answer (D):** There are two cases. If Peter and Pauline are given to the same pet store, then there are 4 ways to choose that store. Each of the children must then be assigned to one of the other three stores, and this can be done in $3^3 = 27$ ways. Therefore there are $4 \cdot 27 = 108$ possible assignments in this case. If Peter and Pauline are given to different stores, then there are $4 \cdot 3 = 12$ ways to choose those stores. In this case, each of the children must be assigned to one of the other two stores, and this can be done in $2^3 = 8$ ways. Therefore there are $12 \cdot 8 = 96$ possible assignments in this case. The total number of assignments is $108 + 96 = 204$.

2010B 15. **Answer (D):** There are three cases to consider.

First, suppose that $i^x = (1+i)^y \neq z$. Note that $|i^x| = 1$ for all x, and $|(1+i)^y| \geq |1+i| = \sqrt{2} > 1$ for $y \geq 1$. If $y = 0$, then $(1+i)^y = 1 = i^x$ if x is a multiple of 4. The ordered triples that satisfy this condition are $(4k, 0, z)$ for $0 \leq k \leq 4$ and $0 \leq z \leq 19$, $z \neq 1$. There are $5 \cdot 19 = 95$ such triples.

Next, suppose that $i^x = z \neq (1+i)^y$. The only nonnegative integer value of i^x is 1, which is assumed when $x = 4k$ for $0 \leq k \leq 4$. In this case $i^x = 1$ and $y \neq 0$. The ordered triples that satisfy this condition are $(4k, y, 1)$ for $0 \leq k \leq 4$ and $1 \leq y \leq 19$. There are $5 \cdot 19 = 95$ such triples.

Finally, suppose that $(1+i)^y = z \neq i^x$. Note that $(1+i)^2 = 2i$, so $(1+i)^y$ is a positive integer only when y is a multiple of 8. Because $(1+i)^0 = 1$, $(1+i)^8 = (2i)^4 = 16$, and $(1+i)^{16} = 16^2 = 256$, the only possible ordered triples are $(x, 0, 1)$ with $x \neq 4k$ for $0 \leq k \leq 4$ and $(x, 8, 16)$ for any x. There are $15 + 20 = 35$ such triples.

The total number of ordered triples that satisfy the given conditions is $95 + 95 + 35 = 225$.

2017B

13. **Answer (D):** By symmetry, there are just two cases for the position of the green disk: corner or non-corner. If a corner disk is painted green, then there is 1 case in which both red disks are adjacent to the green disk, there are 2 cases in which neither red disk is adjacent to the green disk, and there are 3 cases in which exactly one of the red disks is adjacent to the green disk. Similarly, if a non-corner disk is painted green, then there is 1 case in which neither red disk is in a corner, there are 2 cases in which both red disks are in a corner, and there are 3 cases in which exactly one of the red disks is in a corner. The total number of paintings is $1 + 2 + 3 + 1 + 2 + 3 = 12$.